Maskin monotonic coalition formation rules respecting group rights

Koji TAKAMIYA*
Economics, Niigata University

January 7, 2010

Abstract

This paper examines a class of *coalition formation problems* where (i) the list of feasible coalitions (those coalitions which are permitted to form) is given in advance; and (ii) each individual's preference depends only on which coalition this individual belongs to. We are interested in searching for coalition formation rules which are *Maskin monotonic* and respect *coalitional unanimity*. Here a rule is said to respect *coalitional unanimity* if the rule ensures each coalition the "right" of forming that coalition whenever all the members of the coalition rank the coalition at the top. We prove that a rule is Maskin monotonic and respects coalitional unanimity if, and only if, the rule coincides with the strict core stable correspondence. We also consider the implications of this result to Nash implementation and coalition strategy-proofness.

JEL Classification— C71, C72, C78, D02, D71, D78.

Keywords— coalition formation problems, coalitional unanimity, implementation, Maskin monotonicity, strict core stability.

1 Introduction

1.1 Motivation

This paper examines *coalition formation problems* from the viewpoint of mechanism design. In our version of coalition formation problems, the list of the coalitions which are permitted to form are given a priori. These coalitions are called *feasible coalitions*. Each individual is assumed to have a preference ranking over those feasible coalitions which contain this individual. Thus we are ruling out "externalities" in the sense that the people outside the coalition influence the welfare level of the members of the coalition. To our knowledge, our model of coalition formation described above was introduced

*Contact. Faculty of Economics, Niigata University, 8050 2-no-cho Ikarashi Nishi-ku Niigata JAPAN, 950-2181, tel: +81-25-262-6498, E-mail: takamiya@econ.niigata-u.ac.jp.

by Pápai (2004). Our model is general enough to include several important models such as the *marriage problem* and the *roommate problem* (Gale and Shapley, 1962), and the model of *hedonic coalition formation* introduced independently by Banerjee, Konishi and Sönmez (2001), Bogomolnaia and Jackson (2002), and Cechlárová and Romero-Medina (2001).

In this study, we consider *coalition formation rules*, which deterministically specify, for each profile of preferences, a nonempty set of outcomes (formations of coalitions) desirable from the viewpoint of the mechanism designer. Our theme is to identify conclusions from imposing on coalition formation rules the guarantee of *group rights*. This embodies the idea that *each group of people can make a decision on their own at least when the decision is an* internal *concern of the group.* This idea is a natural extension of the idea of "property right" from the individual level to the group level. This study considers a specific form of group right requirement in the following sense: We search for those coalition formation rules which *ensure each feasible coalition the right of forming that coalition at least when all the members of the coalition rank the coalition at the top*. We name this property of coalition formation rules *coalitional unanimity*. Evidently, this is a natural requirement in the context of coalition formation without externalities. This property has been studied in the context of the *marriage problem* by Takagi and Serizawa (2006) and Toda (2006). We find this property favorable from an ethical viewpoint and were motivated to extend this property to a general coalition formation setting.

From the viewpoint of mechanism design, coalition formation rules have to be incentive compatible. We require that coalition formation rules be *Maskin monotonic*. Although Maskin monotonicity itself is not an "incentive property," it is deeply related to various concepts of implementation. For example, as well-known, Maskin monotonicity is a necessary condition for *Nash implementation* (Maskin, 1985, 1999). When the rule is single-valued, on some preferences domains, the property is equivalent to *strategy-proofness* or *coalition strategy-proofness* (Muller and Satterthwaite, 1977; Takamiya, 2007; Bochet and Klaus, 2008). Also it is a necessary and sufficient condition for Nash implementation when mechanisms are allowed to utilize *randomization* (Bochet, 2007; Benoit and Ok, 2008). Further, recent considerations on the *robustness* of implementation to incomplete information have found their relationships with Maskin monotonicity. For instance, some forms of robust implementation in *undominated Nash equilibrium* require social choice rules to be Maskin monotonic whereas under complete information the implementation is possible for non-Maskin monotonic rules. (Chung and Ely, 2003; Kunimoto, 2007).[1] For these known facts, it is highly important to consider Maskin monotonic rules.

[1]In all the results listed above, *deterministic* social choice rules are postulated. However, Maskin monotonicity is also relevant to the *nondeterministic* case: Bochet and Sakai (2005) shows that Maskin monotonicity is a necessary and sufficient condition for the Nash implementation of *stochastic* social choice rules.

1.2 Results

All of our results are proved under some domain assumptions, which generalize *the strict preference domain*, which is commonly assumed in the literature of matching problems. As stated above, our problem is to see what coalition formation rules are Maskin monotonic and respect coalitional unanimity at the same time. Answering this question, we prove that (i) *a rule is Maskin monotonic and respects coalitional unanimity if, and only if, the rule coincides with the strict core stable correspondence.*

Further, we proceed to studying the implications of the above result to *Nash implementation* and *coalition strategy-proofness*: We prove that (ii) *the strict core stable correspondence is Nash implementable if the correspondence is nonempty-valued and there are at least three individuals.* Since Maskin monotonicity is a necessary condition for Nash implementability, we conclude from the two results in the above that (iii) *when there are at least three individuals, a rule is Nash implementable and respects coalitional unanimity if, and only if, the rule coincides with the strict core stable correspondence.* Interestingly, it follows from these results that the Nash implementability problem of the rule which respects coalitional unanimity is reformulated as the *existence problem* of strict core stable partitions.

Regarding *coalition strategy-proofness*, from (i) of our results, we derive the following: (iv) *Given that a rule is* single-valued*, the rule is* coalition strategy-proof *and respects coalitional unanimity if, and only if, the rule coincides with the strict core stable correspondence (thus the strict core stable partition must exist and be unique for all the preference profiles).* This result follows from the conditions for the equivalence of Maskin monotonicity and coalition strategy-proofness given by Takamiya (2007).

1.3 Related works

In the following, we note five papers closely related to our paper. Pápai (2004a) introduces the coalition formation model which we study in this paper, and studies singleton cores under strict preferences. (We note that core stability, which is defined by strong blocking, and strict core stability, which is defined by weak blocking, are equivalent to each other when preferences are strict.) She provides a necessary and sufficient condition, called the *single-lapping condition*, that the set of feasible coalitions is to satisfy for that the set of core stable partitions is a singleton for every profile of strict preferences. Further, she shows that under this condition, the core stable rule is the unique rule which is strategy-proof, individually rational and Pareto efficient.

Pápai's another work (Pápai, 2004b) deals with the existence problem of core stable partitions. It gives a necessary and sufficient condition that the set of feasible coalitions is to satisfy for that the set of core stable partitions is nonempty for every profile of strict preferences.

Takagi and Serizawa (2006) study the *marriage problem* and introduce a property called "pairwise unanimity," which coincides with coalitional unanimity in the context of the marriage problem.[2] They consider single-valued rules and prove that there does not exist any matching rule which is strategy-proof and respects pairwise unanimity.

Toda (2006) also studies marriage problems, and independently introduces a property similar to the pairwise unanimity of Takagi and Serizawa (2006).[3] He proves that any rule which is Maskin monotonic and respects coalitional unanimity is a subcorrespondence of the stable correspondence (which is equivalent to the strict core stable correspondence under strict preferences). This result is generalized in our Theorem 1. Further, Toda obtains characterizations of the stable correspondence using Maskin monotonicity and additional properties.

Takamiya (2008) studies single-valued rules for the same coalition formation model as in the present paper. He proves that if a single-valued rule is strategy-proof and respects coalitional unanimity, then for each preference profile, the set of strict core stable partitions is a singleton or empty and the rule chooses the strict core stable partition whenever available. This result generalizes the above-mentioned impossibility theorem in the marriage problem by Takagi and Serizawa (2006).

2 Preliminaries

2.1 Model of coalition formation

Let $N = \{1, 2, \cdots, n\}$ with $n \geq 2$ be the set of **individuals**. A **coalition** is a nonempty subset of N. A **coalition formation problem** is a list $(N, \mathscr{F}, \succeq)$. Here $\mathscr{F}$ is the set of **feasible coalitions**. $\mathscr{F}$ is a nonempty subset of the set of all coalitions, i.e. $\emptyset \neq \mathscr{F} \subset \{S \mid \emptyset \neq S \subset N\}$. For each $i \in N$, $\mathscr{F}(i)$ denotes the set of feasible coalitions that contain i, i.e. $\mathscr{F}(i) = \{S \mid i \in S \in \mathscr{F}\}$. We assume for any $i \in N$, $\{i\} \in \mathscr{F}$.

A partition of N is called a **feasible partition** if the partition consists only of feasible coalitions. Let x be a feasible partition, and let $i \in N$. Then $x(i)$ denotes the coalition in x which contains i. Let us denote by $X(\mathscr{F})$ the set of feasible partitions. In the following, as long as there is no ambiguity, we refer to them simply "partitions."

$\succeq = (\succeq_i)_{i \in N}$ is a **preference profile**. For each $i \in N$, $\succeq_i$ is a weak ordering (complete and transitive binary relation) over $\mathscr{F}(i)$. As usual, $\succ_i$ denotes the asymmetric part, and $\sim_i$ denotes the symmetric part of $\succeq_i$. Let $x, y \in X(\mathscr{F})$. Then abusing notation, let us denote

$$x \succeq_i y \tag{1}$$

if and only if

$$x(i) \succeq_i y(i) \tag{2}$$

[2] Takagi and Serizawa (2006) also study the *college admission problem*. Refer to Concluding remarks of the present paper.

[3] Toda (2006) calls this property *mutually best*.

Note that we are assuming that preferences are *hedonic*, that is, preferences depend only on the composition of the coalition of which the individual is a member.[4]

As far as we are aware, the present model was introduced by Pápai (2004). This model generalizes the model of *hedonic coalition formation* independently introduced by Banerjee, Konishi and Sönmez (2001), Bogomolnaia and Jackson (2002) and Cechlárová and Romero-Medina (2001), where all coalitions are assumed to be feasible. Also our model includes as special cases the well-known *marriage problems* (two-sided one-to-one matching problems) and *roommate problems* (one-sided one-to-one matching problems) as in Gale and Shapley (1962).

On the other hand, although our *model* also includes *college admission problems* (two-sided many-to-one matching problems), our *results* are not applicable to those problems. This is because in those matching problems, usually preferences are assumed to have some special structures (such as "responsiveness" or "separability"), which are not compatible with the domain assumptions which we will impose later.

2.2 Coalition formation rules

For $i \in N$, $\mathscr{D}_i$ denotes the nonempty set of preference relations admissible to individual i. And $\mathscr{D}$ denotes the domain of preferences, i.e. $\mathscr{D} = \mathscr{D}_1 \times \mathscr{D}_2 \times \ldots \times \mathscr{D}_n$. Given $(N, \mathscr{F}, \mathscr{D})$, a **coalition formation rule** f is a *nonempty set-valued* function $f : \mathscr{D} \to\to X(\mathscr{F})$.

We say that f respects **coalitional unanimity** if for any $\succeq \in \mathscr{D}$ and any $S \in \mathscr{F}$,

$$\big(\forall i \in S,\ \forall T \in \mathscr{F}(i),\ S \succeq_i T\big) \Rightarrow \big(\forall x \in f(\succeq),\ S \in x\big). \tag{3}$$

Note that coalitional unanimity may not be well-defined when preferences permit indifference. This is because in such cases, two coalitions which are both unanimously ranked at the top by their members may have a nonempty intersection. In this case, the definition of the property becomes contradictory. We will impose an assumption (Assumption 2 in Sec.3.1) on the preference domain to exclude such cases and make the condition well-defined.

2.3 Strict core stability

The axiom of coalitional unanimity is closely related to the concept of **strict core stability**. Let a problem $(N, \mathscr{F}, \succeq)$ be given. And let $x \in X(\mathscr{F})$ and $S \in \mathscr{F}$. Then we say that S **blocks** x if

$$\big(\forall i \in S,\ S \succeq_i x(i)\big)\ \&\ \big(\exists j \in S : S \succ_j x(j)\big). \tag{4}$$

A partition x is said to be **strictly core stable** if no feasible coalition blocks x.

[4]The term "hedonic" in this context was coined by Dréze and Greenberg (1980).

The concept of strict core stability is a refinement of **core stability**. It is defined by the stronger notion of blocking which is obtained by replacing the formulae (4) in the above with the following:

$$\forall i \in S,\ S \succ_i x(i). \tag{5}$$

These two core stability concepts are equivalent to each other if preferences are all *strict*, i.e. $S \sim_i T$ implies $S = T$.

The **strict core stable correspondence** is the set-valued function which specifies the set of strict core stable partitions for each preference profile. Let us denote the strict core stable correspondence by $\mathscr{C}$.

2.4 Concepts from mechanism design

Consider a **social choice problem**. The problem consists of (i) the set of **individuals** $N = \{1, \cdots, n\}$ with $n \geq 1$, (ii) the (nonempty) set of **outcomes** A, and (iii) for each $i \in N$, the set of **admissible preferences** over A, which is denoted by $\mathscr{D}_i$. Denote $\mathscr{D} = \mathscr{D}_1 \times \cdots \times \mathscr{D}_n$.

Given a social choice problem, a **social choice rule** (a SCR, for short) is a *nonempty set-valued* function from $\mathscr{D}$ to A. A **game form** (or a **mechanism**) is a list (M, g), where $M = M_1 \times \cdots \times M_n$ and M_i is the **message space** of individual i, and $g : M \to A$ is an **outcome function**. Given a preference profile $\succeq \in \mathscr{D}$, $(M, g, \succeq)$ constitutes a strategic game. Let us denote by $\text{Nash}(M, g, \succeq)$ the set of Nash equilibria of the game $(M, g, \succeq)$. Let a SCR f be given. We say that **a game form** (M, g) **implements the SCR** f **in Nash equilibrium** if for each $\succeq \in \mathscr{D}$, $g(\text{Nash}(M, g, \succeq)) = f(\succeq)$. In this paper, we consider the implementation of *coalition formation rules* as SCRs.

Let a preference relation $\succeq_i$ and an outcome a be given. Then let $\text{MT}(\succeq_i, a)$ denote the set

$$\{\succeq'_i \in \mathscr{D}_i \mid \forall b \in A,\ (a \succeq_i b) \Rightarrow (a \succeq'_i b)\}. \tag{6}$$

For $\succeq \in \mathscr{D}$, denote

$$\text{MT}(\succeq, a) = \{\ \succeq' \in \mathscr{D} \mid\ \forall i \in N,\ \succeq'_i \in \text{MT}(\succeq_i, a)\}. \tag{7}$$

We say that a SCR f is **Maskin monotonic** if for any $\succeq,\ \succeq' \in \mathscr{D}$ and any $x \in f(\succeq)$,

$$\succeq' \in \text{MT}(\succeq, x) \Rightarrow x \in f(\succeq'). \tag{8}$$

It is well-known that Maskin monotonicity is a necessary condition for Nash implementation (Maskin 1985, 1999).

Consider a SCR f. Let us assume f is singleton-valued and regard f as a single-valued function. Then the rule f is **coalition strategy-proof** if for any $\succeq \in \mathscr{D}$, any $S \subset N$, and any $\tilde{\succeq}_S \in \mathscr{D}_S$,

$$\big(\forall i \in S,\ f(\succeq_{-S}, \tilde{\succeq}_S) \succeq_i f(\succeq)\big) \Rightarrow \big(\forall i \in S,\ f(\succeq_{-S}, \tilde{\succeq}_S) \sim_i f(\succeq)\big). \tag{9}$$

There is a strong connection between Maskin monotonicity and coalition strategy-proofness: These two properties are equivalent on the domains considered in the present paper. (See. Sec.3.1.)

3 Results

3.1 Domain assumptions

We define a class of domains by two assumptions. These domains generalize *the strict preference domain*, i.e. the domain such that each $\mathscr{D}_i$ consists *exactly* of those preferences which satisfy

$$\forall i \in N,\ \forall S,\ T \in \mathscr{F}(i),\ S \sim_i T \Rightarrow S = T. \tag{10}$$

The strict preference domain is common in the literature of matching and coalition formation. In the following, we fix the elements $(N, \mathscr{F}, \mathscr{D})$. Correspondingly, let us abbreviate $X(\mathscr{F})$ to X.

Let $\mathscr{Q}$ be a partition of a set Q. Then for any $x \in Q$, let us denote by $\mathscr{Q}(x)$ the cell of $\mathscr{Q}$ that contains x.

Assumption 1 *$\mathscr{D}$ is such that for any $i \in N$, there exists a partition $\mathscr{P}_i$ of X such that*

$$\mathscr{D}_i = \Big\{ \succeq_i \mid \forall x, y \in X,\ \big(x \in \mathscr{P}_i(y) \Leftrightarrow x(i) \sim_i y(i)\big) \Big\}, \tag{11}$$

and

$$\forall x, y \in X,\ \exists i \in N : x \notin \mathscr{P}_i(y). \tag{12}$$

Under Assumption 1, indifferences are permitted; but for each individual i, the indifference class has to be fixed throughout $\mathscr{D}_i$.

The same domain condition as Assumption 1 is found in Takamiya (2007), which calls a domain satisfying this assumption an **essentially strict preference domain**. He has shown that on an essentially strict preference domain, a single-valued rule is Maskin monotonic if, and only if, it is coalition strategy-proof. This result will be applied to obtain our Corollary 2 (Sec.3.2).

Preferences which satisfy Assumption 1 naturally arise when some individual cares only about a part of the composition of the coalition which this individual belongs to.

Note that the indifference class $\mathscr{P}_i$ of X in Assumption 1 uniquely corresponds to the indifference class $\tilde{\mathscr{P}}_i$ of $\mathscr{F}(i)$ in the way that $x \in \mathscr{P}_i(y)$ if and only if $x(i) \in \tilde{\mathscr{P}}_i(y(i))$. Thus in the following, as long as no ambiguity arises, let us equate these two classes and denote them by the same "$\mathscr{P}_i$."

Let a profile of indifference classes $(\mathscr{P}_i)_{i \in N}$ of X be given. And let $\mathscr{D}$ satisfy Assumption 1 with this $(\mathscr{P}_i)_{i \in N}$. Then we impose on $(\mathscr{P}_i)_{i \in N}$ the following assumption. This assumption is in order to make *coalitional unanimity* well-defined.

Assumption 2 *For any $i \in N$ and any $S,\ T \in \mathscr{F}(i)$ with $S \neq T$,*

$$\big(S \in \mathscr{P}_i(T)\big) \Rightarrow \big(\exists j \in S \cap T : S \notin \mathscr{P}_j(T)\big). \tag{13}$$

Note that Assumption 2 implies the following fact which will be used in the proofs of our theorems.

$$\forall i \in N,\ \big\{\{i\}\big\} \in \mathscr{P}_i. \tag{14}$$

That is, no singleton is indifferent to other coalitions.

3.2 Statements of results

All of the following results postulate Assumptions 1 and 2. The following is our main theorem.

Theorem 1 *A coalition formation rule f is Maskin monotonic and respects coalitional unanimity if, and only if, for any $\succeq \in \mathscr{D}$, $f(\succeq) = \mathscr{C}(\succeq)$.*

In the following we apply Theorem 1 to *Nash implementability* and *coalition strategy-proofness.*

Theorem 2 *If $|N| \geq 3$ and the strict core stable correspondence $\mathscr{C}$ is nonempty-valued, then $\mathscr{C}$ is Nash implementable.*

For the case where $|N| = 2$ and $\mathscr{F} = \{\{1\}, \{2\}, \{1, 2\}\}$, the strict core stable correspondence $\mathscr{C}$ is not Nash implementable. This case of the problem is identical with the *marriage problem* (Gale and Shapley, 1962) with one man and one woman. Kara and Sönmez (1996) proves the impossibility of the Nash implementation for this particular case.

Since Maskin monotonicity is necessary for Nash implementation, Theorems 1 and 2 together yield the following result.

Corollary 1 *Let $|N| \geq 3$. A coalition formation rule f is Nash implementable and respects coalitional unanimity if, and only if, for any $\succeq \in \mathscr{D}$, $f(\succeq) = \mathscr{C}(\succeq)$.*

Note that since f is assumed to be nonempty-valued, Corollary 1 implies the existence problem of Nash implementable rules which respect coalitional unanimity is equivalent to the problem of whether strict core stable partitions exist for all preference profiles in the domain. The existence problem of strict core stable partitions has been solved for the case of *the strict preference domain*: Pápai (2004b) gives a necessary and sufficient condition that the set of feasible coalitions is to satisfy for the existence. However, as far as we know, the problem is open for the larger class of domains considered in the present work.

Coalition strategy-proofness is a stringent incentive property for single-valued rules. Under Assumption 1, for single-valued rules, Maskin monotonicity is equivalent to coalition strategy-proofness by the result of Takamiya (2007). (See Sec.3.1.) This yields the following result.[5]

Corollary 2 *Let a coalition formation rule f be single-valued. Then f is coalition strategy-proof and respects coalitional unanimity if, and only if, for any $\succeq \in \mathscr{D}$, $f(\succeq) = \mathscr{C}(\succeq)$.*

Note that Corollary 2 says that $\mathscr{C}$ must be single-valued. A necessary and sufficient condition that the set of feasible coalitions is to satisfy for the single-valuedness of $\mathscr{C}$ on the strict preference domain has been obtained by Pápai (2004a). However, to our knowledge, the problem is also open for the larger class of domains considered in the present work.

In our setting, a coalition formation rule f is defined to be a *non-empty* correspondence. However, since in many cases (such as the *roommate problem* and the *hedonic coalition formation model*) the strict core stable correspondence $\mathscr{C}$ takes empty values. Theorem 1 does not provide any characterization of the correspondence $\mathscr{C}$ in such cases. In order to obtain the characterization of $\mathscr{C}$ regardless of whether it is nonempty-valued or not, we drop the nonempty-valuedness assumption of f and impose on f the "restricted nonemptiness" property defined as follows:

$$\forall \succeq \in \mathscr{D},\ \mathscr{C}(\succeq) \neq \emptyset \Rightarrow f(\succeq) \neq \emptyset. \tag{15}$$

Theorem 3 A correspondence f from $\mathscr{D}$ to X is Maskin monotonic, respects coalitional unanimity and satisfies restricted nonemptiness if, and only if, for any $\succeq \in \mathscr{D}$, $f(\succeq) = \mathscr{C}(\succeq)$.

The proof of Theorem 3 is essentially the same as that of Theorem 1 so we will not present it separately.

Remarks

(1) Theorems 1 and 3 are tight, that is, the three properties, Maskin monotonicity, coalitional unanimity and restricted nonemptiness are independent to each other:

[5] This result appears also in Takamiya (2008) with a different derivation.

(i) The *Pareto correspondence* is Maskin monotonic and satisfies restricted nonemptiness (actually it is nonempty-valued) but does not respect coalitional unanimity.[6]

(ii) Consider the rule f satisfying the following two conditions: (a) for any $\succeq \in \mathscr{D}$ and any $S \in \mathscr{F}$, if S is top-ranked for all the members of S in $\succeq$, then $S \in f(\succeq)$; and (b) for any $\succeq \in \mathscr{D}$ and any $i \in N$, if for any $S \in \mathscr{F}(i)$ there is some $j \in S$ (where it can be $j = i$) for whom S is not top-ranked in $\succeq_j$, then $\{i\} \in f(\succeq)$. Then f respects coalitional unanimity and satisfies restricted nonemptiness (f is nonempty-valued) but is not Maskin monotonic.

(iii) Consider *the correspondence f which assigns the empty set for all preference profiles.* Then f is Maskin monotonic and respects coalitional unanimity but does not satisfy restricted nonemptiness.

(2) As noted, the core stability is equivalent to the *strict* core stability on *strict preference* domains. But on some domains which permit indifference, the *core stable correspondence* is Maskin monotonic but does not respect coalitional unanimity.

(3) Toda (2006) proves that in the context of the *marriage problem*, a special case of the present model, Maskin monotonicity and coalitional unanimity imply $f \subset \mathscr{C}$, a part of our Theorem 1.

(4) Kara and Sönmez (1996) proves the Nash implementability of $\mathscr{C}$ with $|N| \geq 3$ in the context of the *marriage problem*. Our Theorem 2 is a straightforward extension of their result.

4 Proofs

To state our proofs, we need to introduce some notations.

- Let $\succeq$ be a preference profile. Let $x \in X$ and $i \in N$. Then let us define the preference relation $\succeq_i^x$ as follows: If $x(i) = \{i\}$, then $\succeq_i^x = \succeq_i$; otherwise, $\succeq_i^x$ is such that it satisfies the following three conditions:

  (i) $x(i) \succ_i^x \{i\}$,

  (ii) $\nexists S \in \mathscr{F}(i),\ x(i) \succ_i^x S \succ_i^x \{i\}$,

  (iii) $\forall S,\ T \in \mathscr{F}(i) \setminus \{\{i\}\},\ (S \succeq_i T) \Leftrightarrow (S \succeq_i^x T)$.

  Note that the fact (14), which follows from Assumption 2, makes this construction feasible.

[6]The *Pareto correspondence* is the rule f such that for any $\succeq \in \mathscr{D}$,

$$f(\succeq) = \{x \in X \mid \forall y \in X,\ (\forall i \in N,\ y \succeq_i x) \Rightarrow (\forall i \in N,\ y \sim_i x)\}.$$

- Further, let us define the preference relation $\succeq_i^{\uparrow(x)}$ so as to satisfy the following two conditions:

  (i) $\forall S \in \mathscr{F}(i) \setminus \mathscr{P}_i(x(i)),\ x(i) \succ_i^{\uparrow(x)} S$,

  (ii) $\forall S,\ T \in \mathscr{F}(i) \setminus \mathscr{P}_i(x(i)),\ (S \succeq_i T) \Leftrightarrow (S \succeq_i^{\uparrow(x)} T)$.

- Finally, let $\succeq_i^{x\uparrow}$ be a shorthand for $(\succeq_i^x)^{\uparrow(x)}$.

The following Lemma 1 ensures *coalitional unanimity is well-defined* under our assumptions.

Lemma 1 *Let $\succeq \in \mathscr{D}$ and $S^1, S^2 \in \mathscr{F}$. Then if for each $k = 1, 2$,*

$$\forall i \in S^k,\ \big(\forall T \in \mathscr{F}(i),\ S^k \succeq_i T\big), \tag{16}$$

then $S^1 \cap S^2 = \emptyset$.

Proof. Suppose the contrary, that is, there are some $S^1, S^2 \in \mathscr{F}$ for which every member of each coalition ranks that coalition at the top, and $S^1 \cap S^2 \neq \emptyset$. Then for each $i \in S^1 \cap S^2$, $S^1 \in \mathscr{P}_i(S^2)$. This clearly contradicts Assumption 2. □

Proof of Theorem 1. "If" part. Omitted.

"Only if" part. (i) We prove that $\big($ f is Maskin monotonic and respects coalitional unanimity $\big) \Rightarrow \big(f(\succeq) \subset \mathscr{C}(\succeq)\big)$. Suppose the contrary, that is, there exist some $\succeq \in \mathscr{D}$ and $x \in X$ such that $x \notin \mathscr{C}(\succeq)$ and $x \in f(\succeq)$. Then there exists some $S \in \mathscr{F}$ such that S blocks x. Let y be any partition such that $S \in y$. Consider the profile $\tilde{\succeq} = (\succeq_{-S},\ \succeq_S^{\uparrow(y)})$. Then the coalitional unanimity of f implies $\forall z \in f(\tilde{\succeq}),\ S \in z$. On the other hand, since $\tilde{\succeq} \in \mathrm{MT}(\succeq, x)$, the Maskin monotonicity of f implies $x \in f(\tilde{\succeq})$. But obviously $S \notin x$, a contradiction.

(ii) We prove that $\big($ f is Maskin monotonic and respects coalitional unanimity $\big) \Rightarrow \big(f(\succeq) \supset \mathscr{C}(\succeq)\big)$. Let f be Maskin monotonic and respect coalitional unanimity. Then f is *individually rational*, that is, $\forall \succeq \in \mathscr{D},\ \forall x \in f(\succeq),\ \forall i \in N,\ x(i) \succeq_i \{i\}$. This directly follows from the result (i) in the above.

Let $\succeq \in \mathscr{D}$ and $x \in \mathscr{C}(\succeq)$. In the following, we show $x \in f(\succeq)$. Let $y \in X$ with $y \neq x$. And let us classify the coalitions in y into two classes: $\mathscr{S}^0 = \{S \in y \mid \forall i \in S,\ y \sim_i x\}$ and $\mathscr{S}^+ = \{S \in y \mid \exists i \in S,\ x \succ_i y\}$. Then by the fact that no coalition blocks x under $\succeq$ and the latter half of Assumption 1, $\mathscr{S}^0 \cup \mathscr{S}^+ = y$ and $\mathscr{S}^+$ is nonempty.

Consider the profile $\succeq^x$. We show $y \notin f(\succeq^x)$: Suppose $y \in f(\succeq^x)$. Let $T \in \mathscr{S}^+$. If $|T| \geq 2$, then the way that $\succeq^x$ is defined implies $\exists j \in T : \{j\} \succ_j^x y(j)$. Since f is individually rational, $y \notin f(\succeq^x)$ follows, a contradiction. Thus it must be $|T| = 1$, that is, for any $j \in T \in \mathscr{S}^+$ we have $y(j) = \{j\}$. This implies $\forall i \in N, x \succeq_i^x y$, which implies $\succeq^{x\uparrow} \in \text{MT}(\succeq^x, y)$. By Maskin monotonicity, $y \in f(\succeq^{x\uparrow})$. But coalitional unanimity implies $\{x\} = f(\succeq^{x\uparrow})$, a contradiction. Thus we have $y \notin f(\succeq^x)$.

Now we have $\forall y \in X \setminus \{x\}$, $y \notin f(\succeq^x)$. Since f is nonempty-valued, $\{x\} = f(\succeq^x)$.[7] Note that $\succeq \in \text{MT}(x, \succeq^x)$. Maskin monotonicity implies $x \in f(\succeq)$, the desired conclusion. □

Proof of Theorem 2. Our proof is done by checking that $\mathscr{C}$ satisfies the condition of *essential monotonicity* by Yamato (1992). For $a \in X$ and $\succeq_i \in \mathscr{D}_i$, let $L(a, \succeq_i)$ denote the set $\{b \in X \mid a \succeq_i b\}$. Let $Y \subset X$, $y \in Y$ and $i \in N$. And let a rule f be given. Then call y **essential to i in Y with respect to the rule f** if

$$\exists \succeq \in \mathscr{D} : L(y, \succeq_i) \subset Y \ \& \ y \in f(\succeq). \tag{17}$$

Denote these essential elements by $E(f, i, Y)$. f is **essentially monotonic** if for any $\succeq, \tilde{\succeq} \in \mathscr{D}$, and any $x \in f(\succeq)$,

$$\Big(\forall i \in N,\ E(f, i, L(x, \succeq_i)) \subset L(x, \tilde{\succeq}_i)\Big) \Rightarrow x \in f(\tilde{\succeq}). \tag{18}$$

Yamato (1992) shows that given $|N| \geq 3$, f is *Nash implementable* if f is essentially monotonic.[8]

We show that $\forall \succeq \in \mathscr{D}$, $\forall x \in \mathscr{C}(\succeq)$, $\forall i \in N$, $E(\mathscr{C}, i, L(x, \succeq_i)) = L(x, \succeq_i)$. If this is true, then the Maskin monotonicity of $\mathscr{C}$ implies its essential monotonicity. Then since $\mathscr{C}$ is Maskin monotonic, it is essentially monotonic. Let $\succeq \in \mathscr{D}$, $i \in N$ and $x \in \mathscr{C}(\succeq)$. And let $y \in L(x, \succeq_i)$. Then consider a profile $\succeq'$ such that (i) $\forall S \in \mathscr{F}(i) \setminus \mathscr{P}_i(y(i))$, $\big(S \succ_i x(i) \Rightarrow S \succ'_i y(i)\big)$ & $\big(x(i) \succ_i S \Rightarrow y(i) \succ'_i S\big)$, (ii) $y(i) \succeq'_i x(i)$, and (iii) $\forall j \in N \setminus \{i\}$, $\forall S \in \mathscr{F}(j)$, $y(j) \succeq'_j S$. Then $L(y, \succeq'_i) = L(x, \succeq_i)$ and $y \in \mathscr{C}(\succeq')$. Thus y is essential to i in $L(x, \succeq_i)$ w.r.t. $\mathscr{C}$. Since y is arbitrarily taken from $L(x, \succeq_i)$, we have $E(\mathscr{C}, i, L(x, \succeq_i)) = L(x, \succeq_i)$. □

5 Concluding remarks

Here we mention some directions of the further research. (1) Our results have been proved under some domain assumptions, which do not permit special preference structures commonly assumed in *many-to-one matching problems*. Takagi and Serizawa (2006) studies single-valued rules in the context of many-to-one matching problems, and shows an impossibility theorem similar to the one in the *marriage problem* mentioned in Sec.1. They assume *responsive preferences*, and they note that the definition

[7] Actually the restricted nonemptiness of f suffices to obtain $\{x\} = f(\succeq^x)$. Because it is true that $x \in \mathscr{C}(\succeq)$ and $\succeq^x \in \text{MT}(x, \succeq)$, and these imply $\mathscr{C}(\succeq^x)$ contains x thus it is nonempty since $\mathscr{C}$ is Maskin monotonic. Therefore, $f(\succeq^x)$ is nonempty by restricted nonemptiness. This is the only modification we have to make to prove Theorem 3.

[8] Yamato (1992) also shows the converse of this result under an additional domain assumption.

of coalitional unanimity has to be modified to accommodate this domain. It is a theme of the future research to device some domain assumptions to include such domains, and to generalize or modify the definition of coalitional unanimity so that we are able to perform analysis similar to the one done in this paper.[9]

(2) We have proved that the existence problem of the Nash implementable rules which respect coalitional unanimity is paraphrased by the problem of the nonempty-valuedness of the strict core stable correspondence. As mentioned in Sec.3.2, the nonempty-valuedness problem is solved for *the strict preference domain* (Pápai, 2004b) but not for the larger class of domains considered in this paper. We consider this problem worth investigation.

(3) Although it has been proved that the strict core stable correspondence is Nash implementable, no concrete mechanisms have been shown. Our proof of implementability uses the condition developed by Yamato (1992). His proof is based on the construction of an all-purpose mechanism but this mechanism is too complex to be applicable to any practical purposes. It is a problem worth investigation to search for some "simple" mechanisms tailor-made for the present setting. We note that by the result of Tatamitani (2002), no *self-relevant* mechanisms can accomplish this task. Here a mechanism is *self-relevant* if each individual's message space consists only of this individual's own preferences and outcomes.

Acknowledgments I am grateful to Yuji Fujinaka, Ryuichiro Ishikawa, Yoshio Kamijo, Takashi Kunimoto, Ryo Nakajima, Toyotaka Sakai, Ryusuke Shinohara, Tomoichi Shinotsuka, Koichi Suga, Shohei Takagi, Takuma Wakayama, Noaki Watanabe for various comments, discussion and conversation. I also thank the participants of the 21COE-GLOPE Workshop on Social Choice and Game Theory at Waseda University in March 2008 and the seminars at Tsukuba University and Shinshu University. All errors are my own responsibility.

References

[1] Banerjee S, Konishi H, and Sönmez T (2001) Core in a simple coalition formation game, *Soc Choice Welfare* 18: 135–53.

[2] Benoit J-P and Ok EA (2008) Nash implementation without no-veto power, *Games Econ Behavior* 64: 51–67.

[3] Bochet O (2007) Nash implementation with lottery mechanisms, *Soc Choice Welfare* 28: 111–125.

[4] Bochet O and Klaus B (2008) The relation between monotonicity and strategy-proofness, Working paper. Maastricht University.

[9] I thank Shohei Takagi for his comments regarding these points.

[5] Bochet O and Sakai T (2005) Nash implementation in stochastic social choice, Working paper. Maastricht University and Yokohama National University.

[6] Bogomolnaia A and Jackson MO (2002) The stability of hedonic coalition structures, *Games Econ Behavior* 38: 201–30.

[7] Cechlárová K and Romero-Medina A (2001) Stability in coalition formation games, *Int J Game Theory* 29: 487–94.

[8] Chung KS and Ely J (2003) Implementation with near-complete information, *Econometrica* 71: 857-871.

[9] Gale D and Shapley L (1962) College admissions and the stability of marriage, *Amer Math Monthly* 69: 9–15.

[10] Dréze J and Greenberg J (1980) Hedonic coalitions: optimality and stability, *Econometrica* 48: 987–1003.

[11] Kara T and Sönmez T (1996) Nash implementation of matching rules, *J Econ Theory* 68: 425–39.

[12] Kunimoto T (2007) How robust is undominated Nash implementation? Working paper. McGill University.

[13] Maskin E (1985) The theory of implementation in Nash equilibrium: A survey, Hurwicz L, Schmeidler D and Sonnenschein H (eds) *Social Goals and Social Organization.* Cambridge University Press, Cambridge, pp 173–204

[14] Maskin E (1999) Nash equilibrium and welfare optimality, *Rev Econ Stud* 66: pp23–38.

[15] Pápai S (2004a) Unique stability in simple coalition formation games, *Games Econ Behavior* 48: 337–54.

[16] Pápai S (2004b) The existence of stable hedonic coalitions, Working paper. University of Notre Dame.

[17] Roth AE and Sotomayor M (1990) *Two-Sided Matching: A Study in Game Theoretic Modeling and Analysis*, Econometric Society Monograph, vol. 18, Cambridge Univ Press.

[18] Shapley L and Scarf H (1974) On cores and indivisibilities, *J Math Econ* 1: pp23–27.

[19] Sönmez T (1999) Strategy-proofness and essentially single-valued cores, *Econometrica* 67: 677–89.

[20] Takagi S and Serizawa S (2006) An impossibility theorem in matching problems, ISER Discussion Paper Series, No.667.

[21] Takamiya K (2007) Domains of social choice functions on which Maskin monotonicity and coalition strategy-proofness are equivalent, *Econ Letters* 95: 348–54.

[22] Takamiya K (2008) Coalitional unanimity versus strategy-proofness in coalition formation problems, Working paper. Niigata University.

[23] Tatamitani Y (2002) Implementation by self-relevant mechanisms: applications, *Math Soc Sciences* 44: 253–76.

[24] Toda M (2006) Monotonicity and consistency in matching markets, *Int J Game Theory* 34: 13–31.

[25] Yamato (1992) On Nash implementation of social choice correspondences, *Games Econ Behavior* 4: 484-92.