# Corrigendum to "The weak core of simple games with ordinal preferences: implementation in Nash equilibrium" [Games and Economic Behavior 44 (2003) 379–389]

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**Abstract**

We correct an axiom in "The weak core of simple games with ordinal preferences: implementation in Nash equilibrium" [Games and Economic Behavior 44 (2003) 379–389], which does not actually work for the intended results, and retrieve these results.

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*Keywords*— simple game, weak core.

# 1 Introduction

The paper of Shinotsuka and Takamiya [1] contains an axiom which does not actually work for their intended results. This corrigendum presents a corrected version of this axiom and show that those results are retrieved.

First of all, we review the history: Shortly after the publication of [1], Professor Bezalel Peleg at the Hebrew University of Jerusalem kindly notified the authors of [1] (Shinotsuka and Takamiya, who are also two of the present authors) of the error. However, these authors could not fully understand his indication and failed to recognize their mistake. Some years later, one of the present authors (Yu) pointed out the error again presenting a counterexample (Example 1 in this note). At this point, the authors of [1] recognized the error and started to mend the paper.

We express our deep gratitude to Professor Peleg for his kind notice, and the authors of [1] apologize for their blindness. Part of this work has been done while two of the present

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## 2 Definitions

Let $N$ be the nonempty finite set of *individuals*. A *coalition* is a nonempty subset of $N$. Let $\mathscr{W}$ be a nonempty set of coalitions. We call an element of $\mathscr{W}$ a *winning coalition*. Let $X$ be the nonempty set of *alternatives*. Let us call $(N, \mathscr{W}, X)$ a *simple game*.

Let $D_i$ be the set of *admissible preferences* of $i \in N$, whose generic element is denoted by $R_i$. $P_i$ and $I_i$ denote the asymmetric and symmetric parts of $R_i$, respectively. We assume that $D_i$ equals the set of all weak orderings over $X$. For a coalition $S$, $D^S$ denotes the product $\prod_{i \in S} D_i$.[1]

Let $x, y \in X$ and $R \in D^N$, and let $S$ be a coalition. Then we say that $x$ *strongly dominates* $y$ *via* $S$ if $S \in \mathscr{W}$ and for any $i \in S$, it holds $xP_iy$. The *weak core* of the simple game with preferences $(N, \mathscr{W}, X, R)$ is the set of alternatives which are not strongly dominated by any alternatives. Given the data $(N, \mathscr{W}, X, D^N)$, the *weak core correspondence* is the correspondence (i.e. set-valued function) $\mathscr{C} : D^N \rightrightarrows X$ which specifies the weak core of $(N, \mathscr{W}, X, R)$ for each $R \in D^N$.

A *social choice correspondence* (SCC) is a correspondence $\varphi : D^N \rightrightarrows X$ such that for any $R \in D^N$, $\varphi(R) \neq \emptyset$. Although the definition of a SCC incorporates its nonempty-valuedness, we define all axioms in the paper for a general correspondence $\varphi$ from $D^N$ to $X$, which is not necessarily a SCC. This is in order to retain the generality of our results and to clarify under what assumptions the results hold true. In some of our results, instead of assuming that the correspondence is a SCC, we require the satisfaction of the following axiom. Let $\varphi$ be a correspondence $D^N$ to $X$.

**Restricted Nonemptiness (RNEM)** $\forall R \in D^N,\ \mathscr{C}(R) \neq \emptyset \Rightarrow \varphi(R) \neq \emptyset$.

This is a weakening of the assumption that the correspondence is a SCC; that is, if $\varphi$ is a SCC, then RNEM is satisfied trivially. RNEM appears in Section 4 of the original paper [1].

Let us introduce another axiom which is basic for our analysis. For $x \in X$ and $R_i \in D_i$, $L(x, R_i)$ denotes the lower contour set of $x$ at $R_i$, that is, $L(x, R_i) = \{y \in X \mid xR_iy\}$. Let $R_i, R_i' \in D_i$ and $x \in X$. Then we say that $R_i'$ is a *monotonic transformation* of $R_i$ at $x$ if $L(x, R_i) \subset L(x, R_i')$. Denote by $MT(x, R_i)$ the set of all monotonic transformations of $R_i$ at $x$. For $x \in X$ and $R \in D^N$, $MT(x, R)$ denotes the set $\{R' \in D^N \mid \forall i \in N,\ R_i' \in MT(x, R_i)\}$. Now we define the following axiom. Let $\varphi$ be a correspondence $D^N$ to $X$.

[1] In the original paper [1], $D^N$ is assumed to be any domain satisfying some richness condition. But if $X$ is finite, then this richness condition implies that $D^N$ equals the domain that we are assuming here (which is shown in Appendix A of [1]), and we will actually assume the finiteness of $X$ later.

**Maskin Monotonicity (MMON)** $\forall R, R' \in D^N,\ \big(x \in \varphi(R)\ \&\ R' \in MT(x,R)\big) \Rightarrow x \in \varphi(R')$.

We note that (i) $\mathscr{C}$ is not necessarily nonempty-valued; and that (ii) $\mathscr{C}$ satisfies MMON (Lemma 3.1 in [1]).

# 3 Corrected axiom

## 3.1 Error

The erroneous axiom in Shinotsuka and Takamiya [1] is as follows, which appears in Section 2 of [1]. Let $\varphi$ be a correspondence from $D^N$ to $X$.

**Strong Non-Discrimination (SND)** $\forall R \in D^N,\ \forall x,\ y \in X,\ \forall S \subset N,$ $\Big((\forall i \in N \setminus S,\ xI_iy)\ \&\ (S \notin \mathscr{W})\Big) \Rightarrow \Big(x \in \varphi(R) \Leftrightarrow y \in \varphi(R)\Big)$.

Main assertions of [1] are as follows:

(i) It is postulated (without a proof) that *the weak core correspondence $\mathscr{C}$ satisfies SND.*

(ii) Theorem 3.2 states that *if $\varphi$ is a SCC and satisfies MMON and SND, then $\varphi$ is a supercorrespondence of $\mathscr{C}$.* (Additionally, in the proof of Theorem 4.1, it is pointed out that the assumption that $\varphi$ is a SCC can be weakened to that $\varphi$ satisfies RNEM.)

(iii) Corollary 3.6 gives an axiomatization of $\mathscr{C}$ based on the above two results and additional results; *a SCC $\varphi$ is $\mathscr{C}$ if and only if $\varphi$ satisfies MMON, SND and an axiom called "Exclusion".*

(iv) Theorem 4.1 gives another axiomatization, a generalization of Corollary 3.6, where the assumption that $\varphi$ is a SCC is dropped; *a correspondence $\varphi$ is $\mathscr{C}$ if and only if $\varphi$ satisfies MMON, SND, "Exclusion" and RNEM.*

The fundamental error of [1] is found in (i) of the main assertions depicted above; actually, *the weak core correspondence $\mathscr{C}$ does not necessarily satisfy SND.* The following counterexample shows this.

**Example 1** Let $N = \{1,2\}$ and $\mathscr{W} = \{N\}$. Let $R$ be the preference profile such that $yP_1zP_1x$ and $zP_2xI_2y$.

Let $S = \{1\}$. Then if $\mathscr{C}$ satisfies SND, the SND of $\mathscr{C}$ implies $x, y \in \mathscr{C}(R)$ or $x, y \notin \mathscr{C}(R)$. But since $z$ dominates $x$, $\mathscr{C}(R) = \{y, z\}$. Thus $\mathscr{C}$ does not satisfy SND. ■

On the other hand, actually, both the statement and proof of Theorem 3.2 are correct. But because $\mathscr{C}$ does not always satisfy SND, both SND and Theorem 3.2 are meaningless for characterizations of $\mathscr{C}$.

In the rest of this corrigendum, firstly, we present a corrected formulation of SND so that $\mathscr{C}$ satisfies the corrected SND. Secondly, we show that under the corrected axiom, Theorem

3.2 is retrieved. We note that the proof of Theorem 3.2 has to be *entirely* replaced; the former proof does not work at all under the corrected SND. Once these two points have been cleared, the other results (including Corollary 3.6 and Theorem 4.1, the two characterizations of $\mathscr{C}$) are retrieved without substantial change to their proofs except a few minor points to be discussed in Appendix.

### 3.2 Correction

Let us present our *corrected version* of SND. For this purpose, firstly we give one notation. Let $x, y \in X$ and $R_i \in D_i$. Then let us denote

$$xA(R_i)y \tag{1}$$

if the following two conditions are satisfied:

(i) $\nexists z \in X \setminus \{x, y\} : (zI_ix) \vee (zI_iy)$,

(ii) $\nexists z \in X : (yP_izP_ix) \vee (xP_izP_iy)$.

That is, $xA(R_i)y$ means that in the ranking $R_i$, either of the following two is true:

(a) $x$ and $y$ are not indifferent and adjacent to (immediately above or below) each other, and no alternatives other than $x$ or $y$ are indifferent to $x$ or $y$.

(b) $x$ and $y$ are indifferent to each other, and no alternatives other than $x$ or $y$ are indifferent to $x$ or $y$.

Note that if $xA(R_i)y$, then $yA(R_i)x$.

The following is the corrected definition of the axiom in question. Let $\varphi$ be a correspondence from $D^N$ to $X$.

**Corrected Strong Non-Discrimination (c-SND)** $\forall R \in D^N,\ \forall x,\ y \in X,\ \forall S \subset N$,
$$\Big((\forall i \in N \setminus S,\ xI_iy)\ \&\ (S \notin \mathscr{W})\ \&\ (\forall i \in S,\ xA(R_i)y)\Big) \Rightarrow \Big(x \in \varphi(R) \Leftrightarrow y \in \varphi(R)\Big).$$

Compare the above new definition with the original one. The only modification we have made is adding the condition $(\forall i \in S,\ xA(R_i)y)$ to the antecedent. Note that SND implies c-SND.

## 4 Retrieved results

At this point, we introduce two additional assumptions, which were not made in the original paper [1]. These assumptions are indispensable for our remedy and we believe that they are fairly reasonable in themselves. In particular, Assumption 1 is standard in the theory of simple games.

**Assumption 1** *The set of winning coalitions $\mathscr{W}$ is* monotonic, *that is, for any $T, S \subset N$, if $T \in \mathscr{W}$ and $T \subset S$, then $S \in \mathscr{W}$.*

**Assumption 2** *The set of alternatives $X$ is finite.*

First, we show that the corrected axiom clears the point at which the original axiom failed.

**Theorem 1** *The weak core correspondence $\mathscr{C}$ satisfies c-SND.*

*Proof of Theorem 1.* Suppose that the antecedent of c-SND is satisfied, that is, for given $R \in D^N$, $x, y \in X$ and $S \subset N$, the following is satisfied: (i) for any $i \in N \setminus S$, $xI_iy$; (ii) $S \notin \mathscr{W}$; (iii) and for any $i \in S$, $xA(R_i)y$.

Now we will show that $x \notin \mathscr{C}(R)$ implies $y \notin \mathscr{C}(R)$. Suppose that $x \notin \mathscr{C}(R)$. Then there is some alternative $z$ which dominates $x$ via some $T \in \mathscr{W}$, that is, for any $i \in T$, $zP_ix$. By the supposition we have made for $R_i$, it holds $\{i \in N \mid yP_ix\} \subset S \notin \mathscr{W}$. Since $\mathscr{W}$ is monotonic, any subset of $\{i \in N \mid yP_ix\}$ does not belong to $\mathscr{W}$. Thus $y$ does not dominate $x$, that is, $z \neq y$. This implies that $z$ is ranked strictly above also $y$, not only $x$, for each $i \in T$, that is, $z$ dominates $y$ via $T$. (Because for each individual $i \in T$ for whom $yP_ix$, we have $zP_iy$ since $y$ is the only alternative ranked immediately above $x$ and $z \neq y$; and for each individual $i \in T$ for whom $xR_iy$, we have $zP_iy$ simply by the transitivity of $R_i$ and $zP_ix$.) Therefore, $y \notin \mathscr{C}(R)$.

Since $x$ and $y$ are symmetric in the antecedent of c-SND, the same argument as above proves that $y \notin \mathscr{C}(R)$ implies $x \notin \mathscr{C}(R)$. Finally, we conclude $x \in \mathscr{C}(R) \Leftrightarrow y \in \mathscr{C}(R)$. ■

Second, we assert that Theorem 3.2 in [1] is retrieved with the corrected axiom.

**Theorem 2** *Let $\varphi$ be a correspondence from $D^N$ to $X$ which satisfies RNEM. Then if $\varphi$ satisfies MMON and c-SND, then $\varphi \supset \mathscr{C}$.*

The proof of Theorem 2 is somewhat lengthy and presented in the next section.

# 5 Proof of Theorem 2

Let $\varphi$ be a correspondence from $D^N$ to $X$. Then $\varphi$ is said to *respect unanimity* if for any $x \in X$ and any $R \in D^N$, if for any $i \in N$, $L(x, R_i) = X$, then $x \in \varphi(R)$.

**Lemma 1** *Let $\varphi$ be a correspondence from $D^N$ to $X$. Then if $\varphi$ satisfies RNEM, MMON and c-SND, then $\varphi$ respects unanimity.*

*Proof of Lemma 1.* Consider the preference profile $R^0$ in which *any* individual is indifferent between *any* two alternatives. Then since $\emptyset \notin \mathscr{W}$, the antecedent of c-SND are satisfied with $S = \emptyset$ for any two alternatives $x$ and $y$. Thus it follows that $\varphi(R^0)$ treats any two alternatives in the same way, that is, $\varphi(R) = \emptyset$ or $\varphi(R) = X$. But the former is impossible since $\varphi$ satisfies RNEM and $\mathscr{C}(R^0) = X$. Thus we have $\varphi(R^0) = X$.

Now let $x \in X$, and consider any preference profile $R$ such that $L(x, R_i) = X$ for any $i \in N$. Then $R \in MT(x, R^0)$. Since $x \in X = \varphi(R^0)$, MMON implies $x \in \varphi(R)$. Thus $\varphi$

respects unanimity. ■

Before we proceed, we introduce a construction necessary for our proof. Let a linear ordering $\succ$ over $X$ be given, and let $x \in X$ and $Y \subset X \setminus \{x\}$. Then consider the preference $R_i$ of individual $i$ defined as satisfies the following:

(i) $\forall y \in Y,\ yP_ix,$

(ii) $\forall y, y' \in Y,\ (yP_iy' \Leftrightarrow y \succ y'),$

(iii) $\forall y \in X \setminus Y,\ xI_iy.$

That is, this preference is such that all elements in $Y$ are strictly preferred to $x$ and strictly ranked in the same ordering as in $\succ$, and the other alternatives $X \setminus Y$ are all indifferent to $x$. Note that this preference is uniquely determined once $x$, $\succ$, and $Y$ have been given. In the sequel, let us denote this preference by $R_i^{[x,\succ,Y]}$.

**Lemma 2** *Let $\varphi$ be a correspondence from $D^N$ to $X$. And let $\varphi$ satisfy MMON and c-SND. Let $\succ$ be a linear ordering on $X$, let $x \in X$, and for each $i \in N$, let $Y_i \subset X \setminus \{x\}$. And let $y$ be the $\succ$-minimum element of $\bigcup_{i \in N} Y_i$. Then if $x \notin \varphi((R_i^{[x,\succ,Y_i]})_{i\in N})$ and $x \in \mathscr{C}((R_i^{[x,\succ,Y_i]})_{i\in N})$, then $x \notin \varphi((R_i^{[x,\succ,Y_i\setminus\{y\}]})_{i\in N})$.*

*Proof of Lemma 2.* Let $\varphi$ satisfy MMON and c-SND. Let $\succ$ be a linear ordering on $X$, let $x \in X$, and let for each $i \in N$, let $Y_i \subset X \setminus \{x\}$. Let us denote the preference profile $(R_i^{[x,\succ,Y_i]})_{i\in N}$ by $R^\star$. Suppose $x \notin \varphi(R^\star)$ and $x \in \mathscr{C}(R^\star)$. Let $y$ be the $\succ$-minimum element of $\bigcup_{i\in N} Y_i$. Clearly, in the profile $R^\star$, for those individuals who prefer $y$ to $x$, $y$ is ranked immediately above $x$ and for the other individuals, $y$ is indifferent to $x$. Now denote the set of individuals who prefer $y$ to $x$ in $R^\star$ by $T$.

Consider the preference $\tilde{R}_i^\star$ such that for each $i \in T$:

(i) $\forall z \in X \setminus (Y_i \cup \{x\}),\ x\tilde{P}_i^\star z,$

(ii) $\forall z \in Y_i,\ z\tilde{P}_i^\star x,$

(iii) $\forall z, z' \in X \setminus \{x\},\ z\tilde{R}_i^\star z' \Leftrightarrow zR_i^\star z'.$

That is, $\tilde{R}_i^\star$ is obtained by modifying $R_i^\star$ in the way that $x$ is ranked strictly above those alternatives which were indifferent to $x$ in $R_i^\star$ but still $x$ is ranked strictly below $Y_i$, and except $x$, the ranking is unchanged.

Clearly, for each $i \in T$, $\tilde{R}_i^\star \in MT(x, R_i^\star)$ and $R_i^\star \in MT(x, \tilde{R}_i^\star)$. Thus $x \notin \varphi(R_{-T}^\star, \tilde{R}_T^\star)$ by the MMON of $\varphi$ and the supposition $x \notin \varphi(R^\star)$. Now note that in $(R_{-T}^\star, \tilde{R}_T^\star)$ it holds $xA(\tilde{R}_i^\star)y$ for $i \in T$, and $xI_i^\star y$ for $i \in N \setminus T$. And $T \notin \mathscr{W}$ (because otherwise $y$ would dominate $x$ via $T$ in $R^\star$ thus $x \notin \mathscr{C}(R^\star)$. This contradicts the supposition we have made at the start of this proof). Thus all the conditions for the antecedent of c-SND are satisfied. Therefore, the c-SND of $\varphi$ and the fact $x \notin \varphi(R_{-T}^\star, \tilde{R}_T^\star)$ (which we have shown just above) imply $y \notin \varphi(R_{-T}^\star, \tilde{R}_T^\star)$.

Next, consider the preference $\tilde{R}_i^{\star(y)}$ such that for each $i \in T$:

(i) $xP_i^{\star(y)}y$,

(ii) $\forall z \in X \setminus (Y_i \cup \{x\}),\ y\tilde{P}_i^{\star(y)}z$,

(iii) $\forall z \in Y_i \setminus \{y\},\ z\tilde{P}_i^{\star(y)}x$,

(iv) $\forall z, z' \in X \setminus \{x, y\},\ z\tilde{R}_i^{\star(y)}z' \Leftrightarrow z\tilde{R}_i^{\star}z'$.

That is, $\tilde{R}_i^{\star(y)}$ is obtained from $\tilde{R}_i^{\star}$ by flipping the positions of $x$ and $y$ and leaving the positions of the other alternatives unchanged.

Now clearly for each $i \in T$, $\tilde{R}_i^{\star} \in MT(y, \tilde{R}_i^{\star(y)})$. Then since $y \notin \varphi(R_{-T}^{\star}, \tilde{R}_T^{\star})$ as we have shown in the above, the MMON of $\varphi$ implies $y \notin \varphi(R_{-T}^{\star}, \tilde{R}_T^{\star(y)})$. Now note that in $(R_{-T}^{\star}, \tilde{R}_T^{\star(y)})$ it holds $xA(\tilde{R}_i^{\star(y)})y$ for $i \in T$, $xI_i^{\star}y$ for $i \in N \setminus T$, and $T \notin \mathscr{W}$. Thus c-SND implies $x \notin \varphi(R_{-T}^{\star}, \tilde{R}_T^{\star(y)})$.

Now consider the preference $R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]}$ for each $i \in T$. Note that for each $i \in T$, $R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]}$ and $\tilde{R}_i^{\star(y)}$ are the same except the positions of $x$ and $y$, and as for $x$ and $y$ the difference between these two preferences is that in $R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]}$, $x$ and $y$ are indifferent to any element of $X \setminus (Y_i \cup \{x, y\})$ whereas in $\tilde{R}_i^{\star(y)}$, $y$ is ranked strictly and immediately above $X \setminus (Y_i \cup \{x, y\})$ and $x$ is ranked strictly and immediately above $y$. Clearly, for each $i \in T$, $R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]} \in MT(x, \tilde{R}_i^{\star(y)})$ and $\tilde{R}_i^{\star(y)} \in MT(x, R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]})$. Thus the MMON of $\varphi$ implies $x \notin \varphi(R_{-T}^{\star}, R_T^{[x,\ \succ,\ Y_i \setminus \{y\}]})$ (since we have obtained $x \notin \varphi(R_{-T}^{\star}, \tilde{R}_T^{\star(y)})$ at the end of the last paragraph).

For $i \in N \setminus T$, since $y \notin Y_i$ from the first place, it holds $Y_i \setminus \{y\} = Y_i$. Thus $(R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]})_{i \in N} = (R_{-T}^{[x,\ \succ,\ Y_i]}, R_T^{[x,\ \succ,\ Y_i \setminus \{y\}]})$. Recall that $R^{\star} = (R_i^{[x, \succ, Y_i]})_{i \in N}$. Thus $x \notin \varphi(R_{-T}^{\star}, R_T^{[x,\ \succ,\ Y_i \setminus \{y\}]})$ (which we have just shown in the last paragraph) is nothing but $x \notin \varphi((R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]})_{i \in N})$. ■

Now we present the proof of Theorem 2.

*Proof of Theorem 2.* Let $\varphi$ satisfy MMON and c-SND. Consider an arbitrary preference profile $R \in D^N$. If $\mathscr{C}(R) = \emptyset$, then clearly $\varphi(R) \supset \mathscr{C}(R)$. Thus we assume $\mathscr{C}(R) \neq \emptyset$. Let $x \in \mathscr{C}(R)$. In the following we will prove $x \in \varphi(R)$, which establishes $\varphi(R) \supset \mathscr{C}(R)$.

Let a linear ordering $\succ$ over $X$ be given. For each $i \in N$, let $Y_i$ be the set of alternatives $X \setminus L(x, R_i)$. Then let us consider the preference profile $(R_i^{[x, \succ, Y_i]})_{i \in N}$, which we denote $R^{\star}$ in the following. Note that $R^{\star} \in MT(x, R)$ and $R \in MT(x, R^{\star})$. Thus if we have proven $x \in \varphi(R^{\star})$, then by MMON we will conclude $x \in \varphi(R)$, which will complete the proof. So in the following we will prove $x \in \varphi(R^{\star})$.

Our proof is done by contradiction. Suppose $x \notin \varphi(R^{\star})$. Since $\mathscr{C}$ satisfies MMON and $R^{\star} \in MT(x, R)$, we have $x \in \mathscr{C}(R^{\star})$. Thus we can apply Lemma 2 for $R^{\star}$ to obtain $x \notin \varphi((R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]})_{i \in N})$, where $y$ is the $\succ$-minimum element of $\bigcup_{i \in N} Y_i$.

Now note that $(R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]})_{i \in N} \in MT(x, R^{\star})$. So the MMON of $\mathscr{C}$ implies $x \in \mathscr{C}((R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]})_{i \in N})$. Thus we can apply Lemma 2 again this time for $(R_i^{[x,\ \succ,\ Y_i \setminus \{y\}]})_{i \in N}$. Because $X$ is a finite set, clearly we can do this repeatedly and narrow down the set of alternatives ranked above $x$ for all individuals until it becomes the empty set. Therefore, we conclude $x \notin \varphi((R_i^{[x,\,\succ,\,\emptyset]})_{i\in N})$. Note that $(R_i^{[x,\,\succ,\,\emptyset]})_{i\in N}$, $x$ is ranked at the top for all individuals. Now by Lemma 1, $\varphi$ respects unanimity. Thus $x \in \varphi((R_i^{[x,\,\succ,\,\emptyset]})_{i\in N})$. This is a contradiction. Thus we conclude that $x \in \varphi(R^\star)$. This completes the proof. ■

# Appendix

In this appendix, we make some additional corrections. As we noted in Section 3.1, the results in the original paper [1] other than the ones examined in the main text and this appendix are retrieved without substantial change to their proofs.

## A.1 Independence of axioms

The original paper [1] gives (eventually-incorrect) axiomatizations of the weak core correspondence $\mathscr{C}$ using SND, and also shows the independence of the systems of axioms. Although those axiomatizations are retrieved by replacing SND with c-SND, the proofs of the independence are not. Thus here we redo them.

Let us introduce another axiom used in [1]. For $i \in N$, let us denote by $R_i^0$ the preference relation in which all the alternatives are indifferent to $i$. Let $\varphi$ be a correspondence from $D^N$ to $X$.

**Exclusion (EX)** $\forall S \in \mathscr{W},\ \forall R_S \in D^S,\ \Big((\forall i, j \in S,\ R_i = R_j)\ \&\ (\forall i \in S,\ R_i \neq R_i^0)\Big) \Rightarrow \varphi(R_S, R_{-S}^0) \neq X$.

Corollary 3.6 in [1] is an alleged axiomatization of $\mathscr{C}$; a SCC $\varphi$ is $\mathscr{C}$ if and only if $\varphi$ satisfies MMON, SND and EX. In Remarks 3.3 and 3.7 in [1], it is shown that by means of examples, the system of axioms in Corollary 3.6 are independent. Fortunately, when SND is replaced with c-SND, for the independence of c-SND and EX, the two examples in [1] work without modification. But for the independence of MMON, the authors of [1] *mistakenly* assert that "the strong core correspondence" satisfies SND and EX but not MMON.[2] Actually, c-SND is not satisfied by the strong core correspondence, *a fortiori* SND (since SND implies c-SND). Now to show the independence of MMON, we give the following example.

**Example 2** Fix some $i \in N$. And let $\mathscr{W} := \{S \mid i \in S \subset N\}$. Fix some $x \in X$. Then let us define $\varphi$ as follows;

$$\forall R \in D^N,\ \varphi(R) = \{y \in X \mid xI_iy\}. \tag{2}$$

Then $\varphi$ satisfies c-SND and EX but not MMON. Note that this $\varphi$ is a SCC, i.e. nonempty-valued. ■

[2] Let $x, y \in X$, and $R \in D^N$. Then we say that $x$ *weakly dominates* $y$ if for some $S \in \mathscr{W}$, for any $i \in S$, it holds $xR_iy$, and for some $j \in S$, it holds $xP_iy$. The concept of weak domination defines the *strong core* and the *strong core correspondence* in the same way as strong domination defines the weak core and the weak core correspondence.

Theorem 4.1 in [1] is another alleged axiomatization of $\mathscr{C}$, which is a generalization of Corollary 3.6; a correspondence $\varphi$ is $\mathscr{C}$ if and only if $\varphi$ satisfies MMON, SND, EX and RNEM. Note that now the assumption that $\varphi$ is a SCC is dropped, and RNEM is introduced instead. The independence of these four axioms is established by the three examples used in the above and the example given by Remark 4.2 of [1].

## A.2 Lemma 3.10 in [1]

When SND is replaced with c-SND, the original proof of Lemma 3.10 in [1] does not work any more although the result itself still holds true. Thus here we provide a new proof.

We give additional definitions. Let $\varphi$ be a correspondence from $D^N$ to $X$, and $S$ be a coalition. We say that $S$ has *veto power* if there are some $x \in X$ and $R \in D^N$ such that for any $i \in N \setminus S$, it holds $L(x, R_i) = X$, and $x \notin \varphi(R)$.

The following theorem is the retrieval of Lemma 3.10 in [1].

**Theorem 3** *Let $\varphi$ be a correspondence from $D^N$ to $X$. Then if $\varphi$ satisfies RNEM, MMON and c-SND, then for any coalition $S$ which is not winning, $S$ does not have veto power.*

*Proof of Theorem 3.* Let a correspondence $\varphi$ satisfy MMON and c-SND. Let $S$ be a coalition which is not winning. Suppose that $S$ has veto power. Then there are some $x \in X$ and $R \in D^N$ such that for $i \in N \setminus S$, it holds $L(x, R_i) = X$, and $x \notin \varphi(R)$. Now by Theorem 2, $\varphi(R) \supset \mathscr{C}(R)$. This implies $x \notin \mathscr{C}(R)$. Thus there are some winning coalition $T$ and an alternative $y$ such that $y$ strongly dominate $x$ via $T$. But since $x$ is one of the top-ranked alternatives for the members of $N \setminus S$, the coalition $T$ must be a subset of $S$. However, because $S$ is not winning and $\mathscr{W}$ is monotonic by Assumption 1 (recall Section 4), $T$ is not winning either. This is a contradiction. ■

# References

[1] Shinotsuka T and Takamiya K (2003): The weak core of simple games with ordinal preferences: implementation in Nash equilibrium, *Games and Economic Behavior* 44: 379–89.