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On strategy-proofness and essentially single-valued cores: A converse result

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Abstract. In a general model of indivisible good allocation, Sönmez (1999) established that, whenever the core is nonempty for each preference profile, if an allocation rule is strategy-proof, individually rational and Pareto optimal, then the rule is a selection from the core correspondence, and the core correspondence must be essentially single-valued. This paper studies the converse claim of this result. I demonstrate that whenever the preference domain satisfies a certain condition of 'richness', if the core correspondence is essentially single-valued, then any selection from the core correspondence is strategy-proof (even weakly coalition strategy-proof, in fact). In particular, on the domain of preferences in which each individual has strict preferences over his own assignments and there is no consumption externality, such an allocation rule is coalition strategy-proof. And on this domain, coalition strategy-proofness is equivalent to Maskin monotonicity, an important property in implementation theory.

1 Introduction

In a large class of allocation problems with indivisibilities, the concepts of the *core* and *strategy-proofness* of allocation rules are closely related. In his

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recent paper, Sönmez (1999) established that in a general model of indivisible good allocation, whenever the core is nonempty for each preference profile, if an allocation rule is strategy-proof, individually rational and Pareto optimal, then the rule is a selection from the core correspondence, and the core correspondence must be essentially single-valued. (The preference domain is assumed to satisfy some minor conditions.) Here essential single-valuedness means that the correspondence is nonempty-valued and Pareto indifferent (i.e., any two elements of an image of the correspondence are indifferent to all the individuals). Given this sharp conclusion, it is naturally questioned if the converse of this result holds true: Suppose that the core correspondence is essentially single-valued. Then is every selection from the core correspondence strategy-proof? (Individual rationality and Pareto optimality are satisfied automatically.) The objective of this paper is to study this question.

In fact, Sönmez (1999) himself gave a conditional converse of his conclusion: If the core correspondence is essentially single-valued, and the core is externally stable for each preference profile, then any selection from the core correspondence is weakly coalition strategy-proof. Here the external stability of the core means that any (feasible) allocation outside the core is dominated by some allocation in the core. Although the conclusion of weak coalition strategy-proofness is stronger than strategy-proofness, this result needs the 'additional' assumption of external stability. This fact motivates to look for a more direct converse claim. However, a simple counterexample shows that the unconditional converse claim does not hold true. That is, with an appropriate choice of the initial endowments and the set of feasible allocations, there is a domain of preferences on which a selection from the essentially single-valued core correspondence is not strategy-proof. Thus to obtain a direct-type converse of Sönmez's result, it is necessary to impose some conditions on the preference domain.

In this paper, I demonstrate that *whenever the preference domain satisfies a certain condition of 'richness'*, if the core correspondence is essentially single-valued, then any selection from the core correspondence is *strategy-proof*. In fact, such an allocation rule is even *weakly coalition strategy-proof*. Intuitively, the 'richness' condition requires that for any pair of admissible preferences of any individual, a 'mixture' of these preferences be also admissible. In the 'mixture' preference, two allocations (to be specified in advance) keep or improve their relative rankings compared with one of the two 'original' preferences.

In particular, *on the domain of preferences in which each individual has strict preferences over his own assignments and there is no consumption externality*, the unique selection from the single-valued core correspondence is *coalition strategy-proof*. (Essential single-valuedness is nothing but single-valuedness in this case.) I prove this result by showing that on this particular domain, coalition strategy-proofness is equivalent to well-known 'Maskin monotonicity', which plays important roles in implementation theory. This is a generalization of a similar equivalence theorem by Takamiya (2001), proved in the context of *housing markets* (Shapley and Scarf 1974), a special case of the present model.

2 Definitions

The following model is due to Sönmez (1999). This class of allocation problems is a comprehensive economic model with indivisibilities. It includes some well-known classes of problems as subclasses, such as *marriage problems* (Gale and Shapley 1962) and *housing markets* (Shapley and Scarf 1974), and other models.[1] A *generalized indivisible good allocation problem* is a list $(N, \omega, \mathscr{A}^f, R)$. Here N is the (nonempty) finite set of *individuals*. A coalition is a nonempty subset of N. For $i \in N$, $\omega(i)$ denotes the *initial endowment* of individual i. Assume that $\omega(i)$ is a finite set. For $S \subset N$, denote by $\omega(S)$ the set $\bigcup\{\omega(i) \mid i \in S\}$.[2] $\mathscr{A}^f$ is the set of *feasible allocations*. $\mathscr{A}^f$ is a nonempty subset of the set of all allocations $\{x : N \rightarrow\rightarrow \omega(N) \mid \forall a \in \omega(N)\colon \#\{i \in N \mid a \in x(i)\} = 1\}$. Assume $\omega \in \mathscr{A}^f$. $R = (R^i)_{i \in N}$ is a *preference profile*. Here for each $i \in N$, R^i is assumed to be a weak order[3] on $\mathscr{A}^f$. xR^iy reads that to individual i, x is at least as good as y. As usual, I^i and P^i respectively denote the symmetric ('indifferent') and asymmetric ('strictly prefers') parts of R^i. For each individual i, D^i denotes the (nonempty) set of *admissible preferences* of i. For a coalition S, D^S denotes the Cartesian product $\prod_{i \in S} D^i$. Then D^N represents the *domain of preferences*.

Let R^i be a preference relation and $x \in \mathscr{A}^f$. Then $L(x, R^i)$ denotes the lower contour set of x relative to R^i, i.e., $L(x, R^i) := \{y \in \mathscr{A}^f \mid xR^iy\}$. And $L^*(x, R^i)$ denotes the strict lower contour set of x relative to R^i, i.e., $L^*(x, R^i) := \{y \in \mathscr{A}^f \mid xP^iy\}$.

Let a list $(N, \omega, \mathscr{A}^f)$ be given. Fix a preference profile R. Let $x, y \in \mathscr{A}^f$ and S be a coalition. Say that x *dominates* y via S under R if $[x(S) \subset \omega(S)$ & $[\forall i \in S\colon xR^iy]$ & $[\exists j \in S\colon xP^jy]]$. The *core* is the set of all allocations which are not dominated by any other allocation. The *core correspondence* on D^N is the set-valued function $\mathscr{C} : D^N \rightarrow\rightarrow \mathscr{A}^f$ such that for each $R \in D^N$, $\mathscr{C}(R)$ is the core of the problem $(N, \omega, \mathscr{A}^f, R)$. Call $\mathscr{C}$ *essentially single-valued* if $\mathscr{C}$ is nonempty-valued, and Pareto indifferent, i.e. $\forall R \in D^N\colon \forall x, y \in \mathscr{C}(R)\colon \forall i \in N\colon xI^iy$. An allocation x is *Pareto optimal* under R if no other allocation dominates x via N under R.

An *allocation rule* is a function $\varphi : D^N \rightarrow \mathscr{A}^f$. Let $i \in N$, and $R \in D^N$. Then say that individual i manipulates the outcome at preference profile R if

$$\exists R'^i \in D^i\colon \varphi(R^{-i}, R'^i)P^i\varphi(R). \tag{1}$$

Call φ *strategy-proof* if no individual manipulates the outcome at any preference profile. Let S be a coalition. Then say that S manipulates the outcome at preference profile R if

$$\exists R'^S \in D^S\colon [\forall i \in S\colon \varphi(R^{-S}, R'^S)P^i\varphi(R)]. \tag{2}$$

[1] Indeed, in Sönmez (1999), it is discussed that six preceding models are included as subclasses.
[2] Throughout the paper, inclusion '$\subset$' is weak.
[3] A weak order is a complete (thus reflexive) and transitive binary relation.

Call φ *weakly coalition strategy-proof* if no coalition manipulates the outcome at any preference profile. This is stronger a form of nonmanipulability than strategy-proofness. Further, if the concept of coalitional manipulation (2) is weakened to

$$\exists R'^S \in D^S\colon [\forall i \in S\colon \varphi(R^{-S}, R'^S) R^i \varphi(R)] \;\&\; [\exists j \in S\colon \varphi(R^{-S}, R'^S) P^j \varphi(R)], \tag{3}$$

then the corresponding nonmanipulability concept becomes even stronger one called *coalition strategy-proofness*.

Say φ is *Pareto optimal* if for any $R \in D^N$, the allocation $\varphi(R)$ is Pareto optimal under R.

3 Main results

Let a list $(N, \omega, \mathscr{A}^f)$ be given. I impose the following condition on the domain of preferences D^N.

Condition A. Let $i \in N$, $R^i \in D^i$, and $x, y \in \mathscr{A}^f$ be such that yP^ix. Then for all $R'^i \in D^i$, there exists some $R^{*i} \in D^i$ such that

(i) $L^*(x, R^i) \subset L^*(x, R^{*i})$, and $L(x, R^i) \subset L(x, R^{*i})$; and
(ii) $L^*(y, R'^i) \subset L^*(y, R^{*i})$ and $L(y, R'^i) \subset L(y, R^{*i})$.

Note that in the above, for any combination of $\{(R^i, R'^i), (x, y)\}$, such R^{*i} always exists (though not necessarily unique). Thus this condition basically requires that the preference domain be sufficiently 'large'.

Note that given R^i, it is essential to choose x and y so as to satisfy yP^ix. Because otherwise such R^{*i} might not exist. As an intuitive meaning of Condition A, (i) says that x keeps or improves its relative ranking from R^i to R^{*i}, and (ii) says that so does y from R'^i to R^{*i}. So to speak, R^{*i} is a 'mixture' of R^i and R'^i from the viewpoint of desirability of x and y. For example, for each $i \in N$, let $\mathscr{R}_s^i$ denote the set of all preference relations in which individual i has strict preferences over his own assignments (i.e. $xI^iy \Rightarrow x(i) = y(i)$) and there is no consumption externality (i.e. $x(i) = y(i) \Rightarrow xI^iy$). Denote by $\mathscr{R}_s$ the Cartesian product $\prod_{i \in N} \mathscr{R}_s^i$. Then the domain $\mathscr{R}_s$ satisfies Condition A. I will concentrate on this domain in Sect. 4 below.

The following is the main theorem.

Theorem 1. *Assume that the domain D^N satisfies Condition A. Then if the core correspondence $\mathscr{C}$ is essentially single-valued, then any selection from $\mathscr{C}$ is weakly coalition strategy-proof.*

Theorem 1 is an immediate consequence of the two lemmas below.

Lemma 2. *Assume that the core correspondence $\mathscr{C}$ is essentially single-valued. Let φ be a selection from $\mathscr{C}$. Then φ satisfies the property (B) in the following.*

$$\forall R, R^* \in D^N:$$
$$[x = \varphi(R) \;\&\; (\forall i \in N: L^*(x, R^i) \subset L^*(x, R^{*i}) \;\&\; L(x, R^i) \subset L(x, R^{*i}))]$$
$$\Rightarrow [\forall i \in N: \varphi(R^*) I^{*i} x]. \tag{B}$$

Proof. Let $R \in D^N$ and $x = \varphi(R)$. And let $R^* \in D^N$ satisfy

$$\forall i \in N: L(x, R^i) \subset L(x, R^{*i}) \;\&\; L^*(x, R^i) \subset L^*(x, R^{*i}). \tag{4}$$

Since φ is a core-selection, $x \in \mathscr{C}(R)$. This and (4) imply $x \in \mathscr{C}(R^*)$. Then since $\mathscr{C}$ is essentially single-valued, it follows $\forall i \in N$: $\varphi(R^*) I^{*i} x$. ☺

Lemma 3. *Let φ be an allocation rule. Assume that the domain D^N satisfies Condition A. Then φ is weakly coalition strategy-proof if φ is Pareto optimal, and satisfies the property (B).*

Proof. Let φ be Pareto optimal and satisfy the property (B). Suppose that φ is not weakly coalition strategy-proof. Then there is some coalition S which manipulates the outcome (in the sense of the formula (2) in Sect. 2) at a profile R by reporting R'^S. Denote $x = \varphi(R)$ and $y = \varphi(R^{-S}, R'^S)$. Then for any $i \in S$, $yP^i x$. For each $i \in S$, choose a preference relation R^{*i} such that

$$L^*(x, R^i) \subset L^*(x, R^{*i}), \quad \text{and} \quad L(x, R^i) \subset L(x, R^{*i}); \quad \text{and} \tag{5}$$

$$L^*(y, R'^i) \subset L^*(y, R^{*i}) \quad \text{and} \quad L(y, R'^i) \subset L(y, R^{*i}). \tag{6}$$

Condition A assures that D^i includes such R^{*i}. By the property (B), the construction (5) of R^{*S} implies $\forall i \in S$: $\varphi(R^{-S}, R^{*S}) I^{*i} x$ and $\forall i \in N - S$: $\varphi(R^{-S}, R^{*S}) I^i x$. Similarly, by the construction (6), I have $\forall i \in S$: $\varphi(R^{-S}, R^{*S}) I^{*i} y$ and $\forall i \in N - S$: $\varphi(R^{-S}, R^{*S}) I^i y$. Then it follows $\forall i \in S$: $x I^{*i} y$ and $\forall i \in N - S$: $x I^i y$. Now recall that for any $i \in S$, $yP^i x$. Thus under R, y dominates x via N. But $x = \varphi(R)$. This contradicts the Pareto optimality of φ. ☺

4 Further results

Now I restrict my attention to the preference domain $\mathscr{R}_s$ defined above. This domain is remarked in Sönmez (1999) as an important example of the domains compatible with the conditions he imposed on the admissible preferences (which are different from Condition A). By choosing this specific domain, some sharper conclusions become additionally available. First of all, I point out that the property (B) is the same as the following property (C) on the domain $\mathscr{R}_s$.

$$\forall R, R^* \in D^N: [x = \varphi(R) \;\&\; (\forall i \in N: L(x, R^i) \subset L(x, R^{*i}))] \Rightarrow \varphi(R^*) = x. \tag{C}$$

The property (C) is well known as Maskin monotonicity, which plays very important roles in implementation theory (see e.g., Maskin 1985). Note that in general, this property is stronger than the property (B). On the domain $\mathscr{R}_s$, this property is the same as coalition strategy-proofness.[4]

Theorem 4. *Assume that the domain D^N equals $\mathscr{R}_s$. Let φ be an allocation rule. Then φ is coalition strategy-proof if and only if φ satisfies the property (C).*

Proof. (If): Assume φ satisfies the property (C). Suppose that φ is manipulated by some coalition S (in the sense of the formula (3)) at a profile R by reporting R'^S. Denote $x = \varphi(R)$ and $y = \varphi(R^{-S}, R'^S)$. Then for each $i \in S$, choose a preference R^{*i} in $\mathscr{R}_s^i$ for which

(i) no allocation is strictly preferred to y; and
(ii) if an allocation z is strictly preferred to x, then $y(i) = z(i)$.

Note that $\forall i \in S$: $L(x, R^i) \subset L(x, R^{*i})$ and $L(y, R'^i) \subset L(y, R^{*i})$.

Then by the property (C), I have $\varphi(R^{-S}, R^{*S}) = x$ and $\varphi(R^{-S}, R^{*S}) = y$ at the same time. Then I get $x = y$, a contradiction.

(Only If): Assume that φ does not satisfy the property (C). Then I have $\exists R \in D^N$: $\exists i \in N$: $\exists R^{*i} \in D^i$:

$$[x = \varphi(R) \ \& \ L(x, R^i) \subset L(x, R^{*i})] \ \& \ [\exists j \in N: \varphi(R^{-i}, R^{*i})(j) \neq x(j)].$$

Note that since $D^N = \mathscr{R}_s$, $\varphi(R^{-i}, R^{*i})(j) \neq x(j)$ implies that the two allocations $\varphi(R^{-i}, R^{*i})$ and x are never indifferent to j at any profile. Suppose that φ is coalition strategy-proof. Now if $j = i$, then φ would be manipulated by i. Because, when $\varphi(R^{-i}, R^{*i}) P^{*i} \varphi(R)$ $(= x)$, i manipulates at R since $\varphi(R^{-i}, R^{*i}) P^i \varphi(R)$ by the property (C); when $\varphi(R) P^{*i} \varphi(R^{-i}, R^{*i})$, i manipulates at (R^{-i}, R^{*i}). Thus I have $\varphi(R^{-i}, R^{*i})(i) = \varphi(R)(i)$. After all, there must be $j \neq i$ such that $\varphi(R)(j) \neq \varphi(R^{-i}, R^{*i})(j)$. But if $\varphi(R^{-i}, R^{*i}) P^j \varphi(R)$ holds, then (since $\varphi(R^{-i}, R^{*i}) I^i \varphi(R)$) $\{i, j\}$ manipulates at $(R^{-\{i,j\}}, R^j, R^i)$. Similarly, $\varphi(R) P^j \varphi(R^{-i}, R^{*i})$ also implies a manipulation by $\{i, j\}$. This is a contradiction. ☺

Takamiya (2001) proved the same equivalence in the context of *housing markets* (Shapley and Scarf 1974), a special case of the present model. Theorem 4 generalizes this result. Also Theorem 4 is reminiscent of the famous theorem by Muller and Satterthwaite (1977), which states the equivalence of strategy-proofness and Maskin monotonicity for any social choice function defined on the linear (i.e., no indifferences allowed) unrestricted domain of preferences. Readers might find an independent interest in Theorem 4.

From Theorem 4 (together with Lemma 2), I conclude the following. Note that on the domain $\mathscr{R}_s$, essentially single-valuedness is nothing but single-valuedness.

Corollary 5. *Assume that the domain D^N equals $\mathscr{R}_s$. Then if the core correspondence $\mathscr{C}$ is single-valued, then the unique selection from $\mathscr{C}$ is coalition strategy-proof.*

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[4] The following two results have already been reported in Takamiya (1999) in a different context.

5 Example

Finally, to see Condition A is not superfluous to my results, I present an example in which the selection from the essentially single-valued (single-valued, in fact) core correspondence is not strategy-proof. Needless to say, Condition A is violated.

Example 6. Assume $N = \{1, 2, 3\}$:

$$\omega(1) = \{a, b\}, \quad \omega(2) = \{c\}, \quad \omega(3) = \{d\}: \quad \mathscr{A}^f = \{x, y, z, \omega\},$$

where

$$x(1) = \{c, d\}, \quad x(2) = \{a\}, \quad x(3) = \{b\};$$

$$y(1) = \{c, d\}, \quad y(2) = \{b\}, \quad y(3) = \{a\}; \quad \text{and}$$

$$z(1) = \{c\}, \quad z(2) = \{a, b\}, \quad z(3) = \{d\}:$$

$$D^1 = \{R^1, R^{*1}\}, \quad \text{where, } R^1\text{: } (z \;\; P^1 \;\; y \;\; P^1 \;\; x \;\; P^1 \;\; \omega),$$

$$R^{*1}\text{: } (y \;\; P^{*1} \;\; z \;\; P^{*1} \;\; \omega \;\; P^{*1} \;\; x);$$

$$D^2 = \{R^2\}, \quad \text{where } R^2\text{: } (x \;\; P^2 \;\; z \;\; P^2 \;\; y \;\; P^2 \;\; \omega); \quad \text{and}$$

$$D^3 = \{R^3\}, \quad \text{where } R^3\text{: } (x \;\; P^3 \;\; y \;\; P^3 \;\; \omega \;\; P^3 \;\; z).$$

Then $\mathscr{C}(R^1, R^2, R^3) = \{x\}$, and $\mathscr{C}(R^{*1}, R^2, R^3) = \{y\}$.

Denote by φ the unique selection from $\mathscr{C}$. Then it is clear that $\varphi(R^{*1}, R^2, R^3) = yP^1x = \varphi(R^1, R^2, R^3)$. Thus φ is not strategy-proof.

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