Preference revelation games and strong cores of allocation problems with indivisibilities*

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Abstract

This note considers the equilibrium outcomes of the *preference revelation games* in the general model of indivisible good allocation introduced by Sönmez (1999). We adopt the concepts of coalitional equilibria and cores which are both defined in terms of the weak deviation or blocking by a prescribed class of admissible coalitions. We prove that if the solution which induces preference revelation games is *individually rational* and *Pareto optimal* and the class of admissible coalitions is "monotonic," then the set of coalitional equilibrium outcomes coincides with the core. And we point out that the preceding analysis in the context of *marriage problems* (Gale and Shapley, 1962) is hardly extended to the general model.

JEL Classification— C71, C72, C78, D71, D78.

Keywords— generalized indivisible good allocation problem, preference revelation game, strict $\mathcal{G}$-proof Nash equilibrium, $\mathcal{G}$-strong core.

1 Introduction

In this note we consider the equilibrium outcomes of the *preference revelation games* in the general allocation model of indivisible goods introduced by Sönmez (1999). This model includes, as special cases, some well-known problems such as *marriage problems* and *roommate problems* (Gale and Shapley, 1962), *housing markets* (Shapley and Scarf, 1974), and *hedonic coalition formation games* (Banerjee, Konishi and Sönmez (2001) and Bogomolnaia and Jackson (2002)).

In the context of *marriage problems*, the equilibria of preference revelation games have been studied extensively. Among those studies, Sönmez (1997) provides one of the most general results: The set of $\mathcal{G}$*-proof Nash equilibrium* outcomes of the preference revelation games induced by an *individually rational* and *Pareto optimal* solution coincides with the $\mathcal{G}$*-weak core*. Here $\mathcal{G}$ represents a class of coalitions, and these equilibrium and core

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concepts are defined in the fashion that the deviation or blocking power is restricted to the coalitions belonging to $\mathcal{G}$.

In this note we examine in the general setting the relationship between coalitional equilibria and cores similar to Sönmez's result in marriage problems. We find that Sönmez's result is not generalized to the present model, but rather there are significant relationships between the *refined* versions of his coalitional equilibrium and core concepts, namely the *strict $\mathcal{G}$-proof Nash equilibrium* and the *$\mathcal{G}$-strong core*. These versions of the coalitional equilibrium and the core are defined in terms of the *weak* notions of deviation or blocking whereas Sönmez's versions are defined by the strong notions. We prove that with these concepts of the equilibrium and the core, for any $\mathcal{G}$, the equilibrium outcomes are contained in the core, and if $\mathcal{G}$ is *monotonic*, then the set of equilibrium outcomes and the core coincides with each other.[1] As a special case, our results imply that the set of *strict strong Nash equilibrium* outcomes equals the *strong core*.[2]

2 Definitions

2.1 Model

We follow the definitions by Sönmez (1999). A **generalized indivisible good allocation problem** (or a **problem**, simply) is a list $(N, \omega, \mathcal{A}^f, R)$. Here N is the nonempty finite set of **individuals**. A **coalition** is a nonempty subset of N. For $i \in N$, $\omega(i)$ denotes the **initial endowment** of individual i. Each $\omega(i)$ is a finite set. For $S \subset N$, $\omega(S)$ denotes the set $\bigcup\{\omega(i) \mid i \in S\}$.[3] An **allocation** is a correspondence $x : N \twoheadrightarrow \omega(N)$ satisfying (i) for any $i, j \in N$ with $i \neq j$, $x(i) \cap x(j) = \emptyset$, and (ii) $x(N) = \omega(N)$. Here the condition (i) means that no good can be owned by two or more individuals at the same time, and the condition (ii) means no free disposal. Denote the set of all allocations by $\mathcal{A}^0$. $\mathcal{A}^f$ is the set of **feasible allocations**, which is a nonempty subset of $\mathcal{A}^0$ satisfying $\omega \in \mathcal{A}^f$. $R = (R^i)_{i \in N}$ is a **preference profile**. Each R^i is a weak ordering on $\mathcal{A}^f$.[4] xR^iy means that to individual i, x is at least as good as y. As usual, I^i and P^i respectively denote the symmetric ("indifferent") and the asymmetric ("strictly prefers") parts of R^i. For $B \subset \mathcal{A}^f$, $\operatorname{top} R^i(B)$ denotes the set $\{x \in B \mid \forall y \in B : xR^iy\}$.

For each $i \in N$, $\mathcal{D}^i$ denotes the nonempty set of **admissible preferences** of i. For a coalition S, $\mathcal{D}^S$ denotes the Cartesian product $\prod_{i \in S} \mathcal{D}^i$. Let $\mathcal{D}$ denote $\mathcal{D}^N$, the **domain of preferences**. A **solution** is a single-valued function $\varphi : \mathcal{D} \to \mathcal{A}^f$.

One important example of preference domains is the domain $\mathcal{P}$ defined as follows: For each $i \in N$, let $\mathcal{P}^i$ denote the set of all preference relations in which (i) individual i has strict preferences over own assignments (i.e. $xI^iy \Rightarrow x(i) = y(i)$), and (ii) there is no consumption externality (i.e. $x(i) = y(i) \Rightarrow xI^iy$). Let $\mathcal{P}$ denote $\prod_{i \in N} \mathcal{P}^i$. In most of the studies on *marriage problems* (Gale and Shapley, 1962) or *housing markets* (Shapley and Scarf, 1974), the domain $\mathcal{P}$ is assumed.

[1]Here that $\mathcal{G}$ is *monotonic* means that if a coalition S belongs to $\mathcal{G}$, then any supercoalition of S also belongs to $\mathcal{G}$.

[2]The *strict strong Nash equilibrium* is a variant of the strong Nash equilibrium and is defined in terms of the *weak* deviation.

[3]Throughout the paper, inclusion "$\subset$" is weak.

[4]A weak ordering is a complete and transitive binary relation.

2.2 $\mathcal{G}$-weak and -strong cores

Let a problem $(N, \omega, \mathcal{A}^f, R)$ and a nonempty class of coalitions $\mathcal{G}$ be given. Let $x, y \in \mathcal{A}^f$, and let S be a coalition. Then we say that x **weakly dominates** y via S under R if

$$x(S) = \omega(S) \;\&\; \big(\forall i \in S : xR^i y\big) \;\&\; \big(\exists j \in S : xP^j y\big). \tag{1}$$

The **$\mathcal{G}$-strong core** is the set of the allocations which are not weakly dominated by any other allocation via any coalition belonging to $\mathcal{G}$. The **$\mathcal{G}$-strong core correspondence** on $\mathcal{D}$ is the correspondence $\mathcal{C}^{\mathcal{G}} : \mathcal{D} \rightarrow\!\!\rightarrow \mathcal{A}^f$ such that for each $R \in \mathcal{D}$, $\mathcal{C}^{\mathcal{G}}(R)$ is the $\mathcal{G}$-strong core of the problem $(N, \omega, \mathcal{A}^f, R)$. We say that x **strongly dominates** y via S under R if

$$x(S) = \omega(S) \;\&\; \big(\forall i \in S : xP^i y\big). \tag{2}$$

The **$\mathcal{G}$-weak core** is the set of the allocations which are not strongly dominated by any other allocation via any coalition belonging to $\mathcal{G}$. The **$\mathcal{G}$-weak core correspondence** is defined analogously and denoted by $\mathcal{C}^{W\mathcal{G}}$. If $\mathcal{G} = 2^N \setminus \{\emptyset\}$, then the $\mathcal{G}$-strong core ($\mathcal{G}$-weak core, respectively) is called the **strong core** (**weak core**, respectively). Denote the strong core correspondence by $\mathcal{C}$, and the weak core correspondence by $\mathcal{C}^W$.

Call $\mathcal{G}$ **monotonic** if for any two coalitions T and S, if $S \in \mathcal{G}$ and $S \subset T$, then $T \in \mathcal{G}$.

An allocation x is **Pareto optimal** under R if there is no allocation which weakly dominates x via N. And x is **individually rational** under R if for any $i \in N$, $xR^i\omega$. $\mathcal{I}(R)$ denotes the set of individually rational allocations under R. Note that any strong core allocation is both individually rational and Pareto optimal whereas a weak core allocation is individually rational but not necessarily Pareto optimal.

Call a solution φ **Pareto optimal** (**individually rational**, respectively) if for any $R \in \mathcal{D}$, the allocation $\varphi(R)$ is Pareto optimal (individually rational, respectively) under R.

2.3 Preference revelation games

Let a list $(N, \omega, \mathcal{A}^f, \mathcal{D})$ and a solution $\varphi : \mathcal{D} \to \mathcal{A}^f$ be given. Let $R \in \mathcal{D}$. A **preference revelation game** induced by φ is the strategic game $(\mathcal{D}, \varphi, R)$, where each $\mathcal{D}^i$ is the **strategy space** of individual i, and φ is the **outcome function**.

Let a game $(\mathcal{D}, \varphi, R)$ be given. Then a strategy profile $R^\star \in \mathcal{D}$ is a **$\mathcal{G}$-proof Nash equilibrium** if for any $S \in \mathcal{G}$ and any $R'^S \in \mathcal{D}^S$,

$$\big(\forall i \in S : \varphi(R^{\star-S}, R'^S) R^i \varphi(R^\star)\big) \Rightarrow \big(\exists j \in S : \varphi(R^{\star-S}, R'^S) I^j \varphi(R^\star)\big). \tag{3}$$

And $R^\star \in \mathcal{D}$ is a **strict $\mathcal{G}$-proof Nash equilibrium** if for any $S \in \mathcal{G}$ and any $R'^S \in \mathcal{D}^S$,

$$\big(\forall i \in S : \varphi(R^{\star-S}, R'^S) R^i \varphi(R^\star)\big) \Rightarrow \big(\forall i \in S : \varphi(R^{\star-S}, R'^S) I^i \varphi(R^\star)\big). \tag{4}$$

If $\mathcal{G} = 2^N \setminus \{\emptyset\}$, then a $\mathcal{G}$-proof Nash equilibrium (strict $\mathcal{G}$-proof Nash equilibrium, respectively) is called a **strong Nash equilibrium** (**strict strong Nash equilibrium**, respectively). If $\mathcal{G} = \{\{i\} \mid i \in N\}$, then a (strict) $\mathcal{G}$-proof Nash equilibrium is a **Nash equilibrium**.

Denote by $N^{\mathcal{G}}(\mathcal{D}, \varphi, R)$ the set of $\mathcal{G}$-proof Nash equilibria, by $sN^{\mathcal{G}}(\mathcal{D}, \varphi, R)$ the set of strict $\mathcal{G}$-proof Nash equilibria, by $SN(\mathcal{D}, \varphi, R)$ the set of strong Nash equilibria, and by $Nash(\mathcal{D}, \varphi, R)$ the set of Nash equilibria .

3 Results

3.1 Assumptions

In the following, let a list $(N, \omega, \mathcal{A}^f, \mathcal{D})$ be given. In all of our results, we assume that Conditions A and B in the below are satisfied.

Condition A For each $i \in N$, $\mathcal{D}^i$ satisfies the following:

(i) There is no consumption externality, that is, for any $R^i \in \mathcal{D}^i$ and any $x, y \in \mathcal{A}^f$, $x(i) = y(i) \Rightarrow xI^iy$.

(ii) For any $x \in \mathcal{A}^f$, there is some $R^i \in \mathcal{D}^i$ for which
(a) $\operatorname{top} R^i(\mathcal{A}^f) = \{z \mid z(i) = x(i)\}$, and
(b) $\omega(i) \neq x(i) \Rightarrow \operatorname{top} R^i(\mathcal{A}^f \setminus \operatorname{top} R^i(\mathcal{A}^f)) = \{z \mid z(i) = \omega(i)\}$.

Condition B The set $\mathcal{A}^f$ satisfies the following: Let S be a coalition, and let $x, y \in \mathcal{A}^f$. Then if $x(S) = \omega(S) = y(S)$ (which implies $x(N \setminus S) = \omega(N \setminus S) = y(N \setminus S)$), then $\mathcal{A}^f$ contains the allocation v such that $\big(\forall i \in S : v(i) = x(i)\big)$ and $\big(\forall i \in N \setminus S : v(i) = y(i)\big)$.

Note that the domain $\mathcal{P}$ defined in Sec. 2.1 satisfies Condition A. Condition B is satisfied in various problems such as *marriage problems*, *roommate problems* (Gale and Shapley, 1962), *housing markets* (Shapley and Scarf, 1974), and *hedonic coalition formation games* (Banerjee, Konishi and Sönmez (2001) and Bogomolnaia and Jackson (2002)).

3.2 Statements of results

Theorem *Let φ be a solution that is individually rational and Pareto optimal.*
(A) *For any $R \in \mathcal{D}$, $\varphi(sN^{\mathcal{G}}(\mathcal{D}, \varphi, R)) \subset \mathcal{C}^{\mathcal{G}}(R)$.*
(B) *If $\mathcal{G}$ is monotonic, then for any $R \in \mathcal{D}$, $\varphi(sN^{\mathcal{G}}(\mathcal{D}, \varphi, R)) = \mathcal{C}^{\mathcal{G}}(R)$.*

Remarks (1) Clearly if $\mathcal{G} = 2^N \setminus \{\emptyset\}$, then $\mathcal{G}$ is monotonic. Thus it is immediate from our theorem that the set of *strict strong Nash equilibrium* outcomes coincides with the *strong core*. This generalizes the same observation made in the context of *marriage problems* in the preference revelation games induced by *stable solutions* (Shin and Suh, 1996).[5]

(2) In (B) of our theorem, the monotonicity of $\mathcal{G}$ is essential. Without this assumption, the equivalence does not hold even in the context of *marriage problems*, where the equivalence result of the *weak* versions of the coalitional equilibrium and the core holds true. (Sönmez, 1997) (see (i) of (I) in Sec. 3.3). A counter-example is provided in Example 1 in Appendix.

(3) In our theorem, the strict $\mathcal{G}$-proof Nash equilibrium cannot be replaced with the $\mathcal{G}$-proof Nash equilibrium. A counter-example is provided in Example 2(i) in Appendix.

3.3 Extendability of previous results

We note which parts of the previous results in *marriage problems* (other than the one by Shin and Suh (1996) in the above Remark (1)) can be extended to the present model. This is concerned with the relationship between the *$\mathcal{G}$-proof Nash equilibrium* and the *$\mathcal{G}$-weak core* rather than their *strict* or *strong* versions appearing in our Theorem.

(I) In the context of *marriage problems*,

[5] Note that in *marriage problems*, *stable solutions* are equivalent to both *strong core solutions* and *weak core solutions*.

(i) if φ is *individually rational* and *Pareto optimal*, then for any $R \in \mathcal{P}$, $\varphi(N^{\mathcal{G}}(\mathcal{P}, \varphi, R)) = \mathcal{C}^{W\mathcal{G}}(R)$ (Sönmez, 1997). This result yields the two results in (ii) of (I) in the below in the two extreme cases, $\mathcal{G} = \{\{i\} \mid i \in N\}$ and $\mathcal{G} = 2^N \setminus \{\emptyset\}$.

(ii) If φ is a *stable solution*, then for any $R \in \mathcal{P}$, $\varphi(Nash(\mathcal{P}, \varphi, R)) = \mathcal{I}(R)$ (Alcalde, 1996) and $\varphi(SN(\mathcal{P}, \varphi, R)) = \mathcal{C}^W(R)$ (Shin and Suh, 1996).[6]

(II) In our *general model*,

(i) only one direction of inclusion in Sönmez's result is generalized. That is, it holds true that for any $R \in \mathcal{D}$,

$$\varphi(N^{\mathcal{G}}(\mathcal{D}, \varphi, R)) \subset \mathcal{C}^{W\mathcal{G}}(R). \tag{5}$$

This can be proved by the same line as the proof of (A) of our Theorem so we omit the proof.

(ii) The *reverse inclusion* of (5) does not hold in either of the two extreme cases, $\mathcal{G} = \{\{i\} \mid i \in N\}$ and $\mathcal{G} = 2^N \setminus \{\emptyset\}$, even if φ is restricted to *strong core solutions.* (A counter-example is Example 2(ii) in Appendix.) Thus neither of the two results in (ii) of (I) above is generalized to our model.

3.4 Implications

The significance of our results is that they shed light on the consequence of the *manipulation* by misrepresenting preferences. In the present model, Sönmez (1999) and Takamiya (2003) have characterized *non-manipulable* solutions: Given that the strong core is nonempty for each preference profile, a solution is *strategy-proof*, *individually rational* and *Pareto optimal* if, and only if, the solution is a selection from the *strong core correspondence* and the correspondence is *essentially single-valued.*[7] We can draw two implications from our results and the above characterization: (i) In most cases, the strong core correspondence is neither nonempty-valued nor essentially single-valued so it is typical that solutions are open to manipulations. But the consequences of manipulations are again related to strong cores if the solution is individually rational and Pareto optimal. (ii) In the rare cases where the strong core correspondence is both nonempty-valued and essentially single-valued (as in the case of *housing markets*), even if a solution is manipulable, *as a result of manipulations* strong core allocations are likely to be realized if the solution is individually rational and Pareto optimal.[8]

3.5 Proof of Theorem

Proof of (A). Suppose that for some $R \in \mathcal{D}$, $\varphi(sN^{\mathcal{G}}(\mathcal{D}, \varphi, R)) \not\subset \mathcal{C}^{\mathcal{G}}(R)$. Then for some $R' \in \mathcal{D}$, $R' \in sN^{\mathcal{G}}(\mathcal{D}, \varphi, R)$ and $\varphi(R') \notin \mathcal{C}^{\mathcal{G}}(R)$. Thus there is some $x \in \mathcal{A}^f$ which weakly dominates $\varphi(R')$ via some $S \in \mathcal{G}$ under R. Choose $R^{\star S} \in \mathcal{D}^S$ satisfying for each $i \in S$,

$$\operatorname{top} R^{\star i}(\mathcal{A}^f) = \{z \mid z(i) = x(i)\}, \text{ and} \tag{6}$$

$$\omega(i) \neq x(i) \Rightarrow \operatorname{top} R^{\star i}(\mathcal{A}^f \setminus \operatorname{top} R^{\star i}(\mathcal{A}^f)) = \{z \mid z(i) = \omega(i)\}. \tag{7}$$

[6] Note that Shin and Suh (1996) provides two distinct results, the one in Remark (1) in Sec. 3.2 and the other one stated here.

[7] The strong core correspondence is *essentially single-valued* if for each preference profile any two allocations in the strong core are Pareto indifferent.

[8] Recently Bochet and Sakai (2007) have made a similar observation in the *division problem with single-peaked preferences* (Sprumont, 1991). In this problem, it is known that the *uniform rule* is strategy-proof. They show that even if the solution is not strategy-proof, any strong Nash equilibrium outcome coincides with the *uniform rule allocation* if the solution satisfies some regular properties.

Such $R^{\star S}$ exists in $\mathcal{D}^S$ by Condition A. Since φ is individually rational, for each $i \in S$, $\varphi(R'^{-S}, R^{\star S})(i)$ equals $x(i)$ or $\omega(i)$.

Denote $\varphi(R'^{-S}, R^{\star S})$ by y. Now we prove $\forall i \in S : x(i) = y(i)$. Suppose that for some $i \in S$, $x(i) \neq y(i)$ (which implies $y(i) = \omega(i) \neq x(i)$). Since x weakly dominates $\varphi(R')$ via S under R, it holds true that $x(S) = \omega(S)$. And $y(S) = \omega(S)$ since $\forall i \in S : y(i) = x(i)$ or $= \omega(i)$. Then consider an allocation v which satisfies

$$\forall i \in S : v(i) = x(i), \text{ and} \tag{8}$$

$$\forall i \in N \setminus S : v(i) = y(i). \tag{9}$$

Condition B ensures that $\mathcal{A}^f$ contains this allocation v. Then v Pareto-dominates y. But this contradicts the Pareto optimality of φ. Therefore, it holds true that $\forall i \in S : x(i) = y(i)$. Recall that x weakly dominates $\varphi(R')$ via S under R. Then it follows $\big(\forall i \in S : yR^i\varphi(R')\big)$ & $\big(\exists j \in S : yP^j\varphi(R')\big)$. This implies $R' \notin sN^{\mathcal{G}}(\mathcal{D}, \varphi, R)$, a contradiction. □

Proof of (B). It suffices to prove that $\forall R \in \mathcal{D}$, $\mathcal{C}^{\mathcal{G}}(R) \subset \varphi(sN^{\mathcal{G}}(\mathcal{D}, \varphi, R))$ under the monotonicity of $\mathcal{G}$. Let $R \in \mathcal{D}$ and $x \in \mathcal{C}^{\mathcal{G}}(R)$. Choose $R' \in \mathcal{D}$ such that for any $i \in N$,

$$\operatorname{top} R'^i(\mathcal{A}^f) = \{z \mid z(i) = x(i)\}, \text{ and} \tag{10}$$

$$\omega(i) \neq x(i) \Rightarrow \operatorname{top} R'^i(\mathcal{A}^f \setminus \operatorname{top} R'^i(\mathcal{A}^f)) = \{z \mid z(i) = \omega(i)\}. \tag{11}$$

Such R' exists in $\mathcal{D}$ by Condition A. Then since φ is Pareto optimal, $\varphi(R') = x$.

Now we prove $R' \in sN^{\mathcal{G}}(\mathcal{D}, \varphi, R)$ (thus $x \in \varphi(sN^{\mathcal{G}}(\mathcal{D}, \varphi, R))$). Suppose the contrary. Then there is some coalition $S \in \mathcal{G}$ and $R^{\star S} \in \mathcal{D}^S$ such that

$$\big(\forall i \in S : \varphi(R'^{-S}, R^{\star S})R^i\varphi(R')\big) \;\&\; \big(\exists j \in S : \varphi(R'^{-S}, R^{\star S})P^j\varphi(R')\big). \tag{12}$$

Denote $\varphi(R'^{-S}, R^{\star S})$ by y. Then if $y(S) = \omega(S)$, then (12) implies that y weakly dominates x via S under R, which says $x \notin \mathcal{C}^{\mathcal{G}}(R)$, a contradiction. Thus it must be $y(S) \neq \omega(S)$. Now let T denote an $\subset$-minimal element of the class of coalitions $\{G \subset N \mid S \subset G \;\&\; y(G) = \omega(G)\}$. Thus for any $i \in T \setminus S$, $y(i) \neq \omega(i)$. (Otherwise, T is not $\subset$-minimal.) Then since φ is individually rational, the construction of R' implies $\forall i \in T \setminus S : x(i) = y(i)$, which implies xI^iy. This and (12) together say $(\forall i \in T : yR^ix)$ and $(\exists j \in S \subset T : yP^jx)$, and by construction $y(T) = \omega(T)$. This says that y weakly dominates x via T under R. Since $\mathcal{G}$ is monotonic, $T \in \mathcal{G}$. Thus it follows $x \notin \mathcal{C}^{\mathcal{G}}(R)$, a contradiction. This concludes $R' \in sN^{\mathcal{G}}(\mathcal{D}, \varphi, R)$. □

Appendix

Here we provide the counter-examples in connection with Remarks (2), (3) in Sec. 3.2 and (II) in Sec. 3.3.

Example 1 This example shows that *even in the marriage problem, without the monotonicity of $\mathcal{G}$, (B) of Theorem does not hold true.* (Remark (2) in Sec. 3.2.)

Consider the *marriage problem* (Gale and Shapley, 1962) with two men and two women: $N = M \cup W$ where $M = \{1, 2\}$ and $W = \{3, 4\}$; $\forall i \in N : \omega(i) = \{i\}$; and

$$\mathcal{A}^f = \Big\{x \in \mathcal{A}^0 \mid \big(\forall i \in N : \#x(i) = 1\big) \;\&\; \big(\forall m \in M : x(m) \subset W \cup \{m\}\big) \;\&\; \big(\forall w \in W : x(w) \subset M \cup \{w\}\big) \;\&\; \big(\forall i, j \in N : x(i) = \{j\} \Leftrightarrow x(j) = \{i\}\big)\Big\}. \tag{13}$$

The domain $\mathcal{D}$ equals $\mathcal{P}$.[9] Let φ be a stable solution. Let $\mathcal{G}$ be $\{\{i\} \mid i \in N\} \cup \{\{1, 2, 3\}\}$. Clearly, $\mathcal{G}$ is not monotonic.

[9]In this case, a preference relation on the feasible allocations (matchings) is identified with the corresponding relation on the set of men (women, respectively) plus herself (himself, respectively).

Consider $R \in \mathcal{P}$ defined in the sequel, for which $\mathcal{C}^{\mathcal{G}}(R) \not\subset \varphi(sN^{\mathcal{G}}(\mathcal{P}, \varphi, R))$: Let $R \in \mathcal{P}$ be such that

$$3P^1 1P^1 4, \tag{14}$$
$$4P^2 2P^2 3, \tag{15}$$
$$1P^3 3P^3 2, \tag{16}$$
$$2P^4 4P^4 1. \tag{17}$$

Consider the matching $x \in \mathcal{A}^f$ such that

$$x(1) = 1, \tag{18}$$
$$x(2) = 4, \tag{19}$$
$$x(3) = 3. \tag{20}$$

It is easy to show $x \in \mathcal{C}^{\mathcal{G}}(R)$. But $x \notin \varphi(sN^{\mathcal{G}}(\mathcal{P}, \varphi, R))$, which is proved as follows: Let $R' \in \mathcal{P}$ be any profile such that $x = \varphi(R')$. We prove $R' \notin sN^{\mathcal{G}}(\mathcal{P}, \varphi, R)$. Consider the profile $(R^{\{1,2,3\}}, R'^4)$. Under this profile, there is the unique stable matching y such that

$$y(1) = 3, \tag{21}$$
$$y(2) = 4. \tag{22}$$

(This is because: (i) It is clear that 1 and 3 have to be matched with each other under $(R^{\{1,2,3\}}, R'^4)$. (ii) If 4 were not matched with 2 under $(R^{\{1,2,3\}}, R'^4)$, then 4 would have to be matched with herself. But this implies that 4 would have been matched with herself under R'.) Then $y = \varphi(R^{\{1,2,3\}}, R'^4)$. Note that yP^1x, yP^3x and yI^2x. Thus $R' \notin sN^{\mathcal{G}}(\mathcal{P}, \varphi, R)$, the desired conclusion. We conclude $\mathcal{C}^{\mathcal{G}}(R) \not\subset \varphi(sN^{\mathcal{G}}(\mathcal{P}, \varphi, R))$.

Example 2 This example shows that (i) *the $\mathcal{G}$-proof Nash equilibrium is not applicable to Theorem* (Remark (3) in Sec. 3.2) and (ii) *the reverse inclusion of the formulae (5) does not hold in either of the two extreme cases of $\mathcal{G}$* ((ii) of (II) in Sec. 3.3).

Consider the *housing market* (Shapley and Scarf, 1974) with three traders: $N = \{1, 2, 3\}$; $\forall i \in N : \#\omega(i) = 1$; and $\mathcal{A}^f = \{z \in \mathcal{A}^0 \mid \forall i \in N : \#z(i) = 1\}$. The domain $\mathcal{D}$ equals $\mathcal{P}$.[10] In this case, the strong core correspondence is singleton-valued (Roth and Postlewaite, 1977). Denote $\omega(i)$ simply by i. Let φ be the strong core solution.[11]

(i) Consider $R \in \mathcal{P}$ defined in the sequel, for which $\varphi(SN(\mathcal{P}, \varphi, R)) \not\subset \mathcal{C}(R)$: Let $R \in \mathcal{P}$ be such that

$$2P^1 1P^1 3, \tag{23}$$
$$3P^2 1P^2 2, \tag{24}$$
$$1P^3 3P^3 2. \tag{25}$$

Note that $\varphi(R) = (2, 3, 1)$. Let $x = (2, 1, 3)(\neq \varphi(R))$. And choose another profile $R' \in \mathcal{P}$ such that

$$3P'^1 2P'^1 1, \tag{26}$$
$$1P'^2 2P'^2 3, \tag{27}$$
$$3P'^3 1P'^3 2. \tag{28}$$

[10] In this case, a preference relation on $\mathcal{A}^f$ is identified with the corresponding relation defined on $\omega(N)$.

[11] In this example, abusing notation, for an allocation x, x denotes the triple $(x(1), x(2), x(3))$.

Then $\varphi(R') = x$. And it is easy to show $R' \in SN(\mathcal{P}, \varphi, R)$. Thus one has $x \in \varphi(SN(\mathcal{P}, \varphi, R))$ and $x \notin \mathcal{C}(R)$, i.e. $\varphi(SN(\mathcal{P}, \varphi, R)) \not\subset \mathcal{C}(R)$. (Also, this implies $\varphi(Nash(\mathcal{P}, \varphi, R)) \not\subset \mathcal{C}(R)$.)

(ii) Consider $R \in \mathcal{P}$ defined in the sequel, for which $\mathcal{C}^W(R) \not\subset \varphi(Nash(\mathcal{P}, \varphi, R))$: Let $R \in \mathcal{P}$ such that

$$2P^1 1 P^1 3, \tag{29}$$
$$1P^2 3 P^2 2, \tag{30}$$
$$1P^3 3 P^3 2. \tag{31}$$

First, note that $x = (2, 3, 1) \in \mathcal{C}^W(R)$. Let $R' \in \mathcal{P}$ be any profile such that $x = \varphi(R')$. Then by the well-known "top trading cycle" algorithm for computing strong core allocations (Shapley and Scarf, 1974), it can be shown that any such R' satisfies $\forall i \in N : x(i)$ ranks top in R'^i. Now choose any $R^{\star 2} \in \mathcal{P}^2$ which ranks the good 1 as the best. Then $\varphi(R'^{-2}, R^{\star 2}) = (2, 1, 3)$. Now clearly, $\varphi(R'^{-2}, R^{\star 2})(2)P^2\varphi(R')(2)$. This says that any such R' cannot be a Nash equilibrium of the game $(\mathcal{P}, \varphi, R)$. Thus one has $x \in \mathcal{C}^W(R)$ and $x \notin \varphi(Nash(\mathcal{P}, \varphi, R))$, i.e. $\mathcal{C}^W(R) \not\subset \varphi(Nash(\mathcal{P}, \varphi, R))$. This implies $\mathcal{I}(R) \not\subset \varphi(Nash(\mathcal{P}, \varphi, R))$ and $\mathcal{C}^W(R) \not\subset \varphi(SN(\mathcal{P}, \varphi, R))$.

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