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The consistency principle and an axiomatization of the α-core*

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Abstract. This paper examines the α-core of strategic games by means of the consistency principle. I provide a new definition of a reduced game for strategic games. And I define consistency (CONS) and two forms of converse consistency (COCONS and COCONS*) under this definition of reduced games. Then I axiomatize the α-core for families of strategic games with bounded payoff functions by the axioms CONS, COCONS*, weak Pareto optimality (WPO) and one person rationality (OPR). Furthermore, I show that these four axioms are logically independent. In proving this, I also axiomatize the α-individually rational solution by CONS, COCONS and OPR for the same families of games. Here the α-individually rational solution is a natural extension of the classical 'maximin' solution.

JEL Classification: C71, C72.

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0. Introduction

This paper examines the α-core of strategic games from an axiomatic point of view. The α-core is one of several core concepts in strategic games. The fol-

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lowing scenario describes a main idea of the concept. Consider a strategic game. Assume that a strategy profile is 'suggested' for play. Each coalition has a chance to deviate from the suggested strategies. In other words, each coalition may '*block*' the suggested strategies. In blocking the strategy profile, a coalition must select their blocking strategies firstly. And secondly, the remaining players move in response. And a coalition must 'improve its position' by blocking. The problem is how one defines this 'improvement'. The α-core assumes that a coalition has a 'pessimistic perception[1]' about the choices by the complementary coalition. That is, a coalition blocks a strategy profile if each member of the coalition strictly improves upon the 'suggested' payoff, *no matter what strategies* the remaining players choose. The α-core is the set of all strategy profiles which are not blocked by any coalition. The α-core was introduced by Aumann (1961). Scarf (1971) proved the existence of the α-core for a general class of games. Later, Kajii (1992) and Yannelis (1991) investigated the existence problem further.

This paper adopts an axiomatic approach. One of the purposes of this method is to compare one solution with another. Especially in this paper, the *consistency principle* plays a main role. *Consistency* is a general property that many ideas appearing in social sciences have in common. It has been shown that a considerable number of concepts from allocation rules, public finance, game theory and many other fields in social sciences satisfy consistency. The property gives a beautiful and strong unity across a variety of distinct subjects[2]. This is the reason why we call it a 'principle', not merely a 'property'. Particularly in game theory, many solutions have been axiomatized by consistency. The following describes a general idea of consistency in game theory[3]: Let G be a game[4]. Let x be an outcome which 'solves' the game. Assume that some of the players of the game leave the scene and the remaining players, denoted by S, play the '*reduced*' game $G^{S,x}$. Apply the same solution to $G^{S,x}$. The solution is said to satisfy consistency if $x|S$, the restriction of x to S, results as a solution outcome of $G^{S,x}$. It is not always easy to find how the reduced games $G^{S,x}$ should be defined. For coalitional form games, several definitions of reduced games are available, while only one is available for strategic form games to the best of my knowledge.

In the theory of coalitional form games[5], research on the consistency principle has a long history. And a large literature has been published. On the other hand, studies on the consistency principle for strategic games is relatively new. As far as I know, the first systematic research in this area is Peleg and Tijs (1996)(P&T, henceforth). They defined reduced games in strategic form. And they gave axiomatic characterizations to the following solutions for strategic games: Nash equilibria (Nash, 1951), strong Nash equilibria (Aumann, 1959), semi-strong Nash equilibria (Kaplan, 1992) and coalition-proof Nash equilibria (Bernheim, Peleg and Whinston, 1987). All these axiomatizations are done by

[1] I borrowed this terminology from Ichiishi (1997).
[2] For comprehensive survey, see Thomson (1990, 1996).
[3] The following description on consistency is due to Aumann (1987).
[4] G may take any form, TU, NTU or strategic.
[5] I avoid the term 'cooperative game theory' in this context on the grounds that what distinguishes between non-cooperative and cooperative games is not game forms, but the restriction on the behavior of players. Indeed, the α-core is a cooperative solution in strategic forms. I owe this viewpoint to Ichiishi (1997).

consistency together with various forms of its converse, *converse consistency*. Peleg, Potters and Tijs (1996) explored some general relations between consistency and the non-emptiness axiom using a graph theoretical method. Norde, Potters, Reijnierse and Vermeulen (1996) gave further axiomatizations of Nash equilibria on two classes of strategic games: The class of mixed extensions of finite games and the class of games with continuous concave payoff functions. It is worth noting that their characterizations do not employ any form of converse consistency. All these studies are based on the reduced games introduced in P&T (PT reduced games, henceforth).

Unfortunately, there are solutions to strategic games which violate consistency under PT reduced games. The α-core, defined on some classes of strategic games, falls into this category. This raises the questions: How should I define the reduced game concept to restore consistency for the α-core? And then with what other axioms can I axiomatize the α-core by consistency? These are the problems that this paper attempts to answer to. Precisely, I shall provide an axiomatization of the α-core correspondence on any *closed*[6] family of games (in strategic form) *with bounded payoff functions* by consistency under a *new definition of a reduced game*. Further, I shall show the logical independence of the axioms used in the axiomatization. In proving this, I also axiomatize the α-individually rational solution by consistency for the same families of games as the ones on which the α-core is axiomatized. Here the α-individually rational solution is a natural extension of the classical 'maximin' solution.

The plan of the paper is as follows: The next section introduces some preliminaries. Section 2 provides some definitions, including a new definition of reduced games. Section 3 states the main results with some lemmas. Finally, in Section 4, I discuss the interpretations of the new reduced games. Throughout, I will compare the notions and results from P&T with mine.

1. Preliminaries

In this section, I introduce some basic definitions.

(1) *Strategic form game*: A *strategic* (*form*) *game*, is a list $G := (N(G), \{X_j, u_j\}_{j \in N(G)})$. Here $N(G)$ is the finite set of players. X_j is the (non-empty) strategy set of player $j \in N(G)$. For a coalition S, $S \subset N(G)$[7] with $S \neq \emptyset$, X_S denotes the Cartesian product $\prod_{j \in S} X_j$[8]. And $u_j : X_{N(G)} \to \Re$[9] is the payoff function of player $j \in N(G)$. For $S \subset N(G)$ with $S \neq \emptyset$, $x_S \in X_S$ and $j \in N(G)$, denote by $\mathrm{Im}(u_j|_{x_S})$ the image $u_j(\{x_S\} \times X_{N(G)\setminus S})$, i.e., the set $\{u_j(x_S, y_{N\setminus S}) \mid y_{N\setminus S} \in X_{N(G)\setminus S}\}$.

(2) *The α-core*: Let a strategic game G be given. Let $S \subset N(G)$ with $S \neq \emptyset$, and $x \in X_{N(G)}$. Say that a coalition S *blocks* a strategy profile x if $\exists y_S \in X_S$: $\forall j \in S$: $\inf \mathrm{Im}(u_j|_{y_S}) > u_j(x)$. A strategy profile x is said to be in the *α-core* of G if no coalition blocks x. Thus the α-core is the set of all strategy profiles which are immune to blocking. Note that my definition of block-

[6] See Definition 2-2 in Section 2.
[7] Throughout the paper, inclusion '$\subset$' is weak.
[8] This subscript notation applies also for a strategy profile, e.g. x_S denotes $(x_j)_{j \in S} \in X_S$.
[9] $\Re$ denotes the real line.

ing is slightly stronger than the 'standard' definition: Most authors define that S blocks x if $\exists y_S \in X_S$: $\forall y_{N\setminus S} \in X_{N\setminus S}$: $\forall j \in S$: $u_j(y_S, y_{N\setminus S}) > u_j(x)$. Thus my α-core is slightly larger than the 'standard' one. All preceding existence results (e.g. Scarf, 1971) remain true for my definition. For further remark on this modification, see Remark 3-4.[10]

(3) *The α-individually rational solution*: A strategy profile x is said to be *α-individually rational* if no singleton coalition blocks x. The *α-individually rational solution* (α-IR solution) of a strategic game G is the set of all α-individually rational strategy profiles. The definition of α-individual rationality is a natural extension of the traditional 'maximin' type individual rationality. It is easy to see that a strategy profile x is α-individually rational if and only if for each player j, x gives him a payoff at least as large as his 'security level', i.e., $u_j(x) \geq \sup_{\xi_j} \inf_{\xi_{N\setminus\{j\}}} u_j(\xi_j, \xi_{N\setminus\{j\}})$. By definition, the α-IR solution contains the α-core.

2. Definitions

This section provides some definitions. Henceforth, without mentioning, I refer to a strategic game simply as a game.

Definition 2-1. *Let $G = (N(G), \{X_j, u_j\}_{j\in N(G)})$ be a game. Let $x \in X_{N(G)}$ and $S \subset N(G)$ with $S \neq \emptyset$. Then the* reduced game of G with respect to S and x *is a game $G^{S,x} = (S, \{X_j, u_j^{S,x}\}_{j\in S})$, where the payoff functions $u_j^{S,x}$ are defined as follows:*

$$u_j^{S,x}(y_S) = \inf \operatorname{Im}(u_j|_{y_S}), \quad \text{if } y_S \neq x_S,$$
$$= u_j(y_S, x_{N\setminus S}) \quad \text{otherwise}, \quad \text{for any } y_S \in X_S.$$

Some discussion on reduced games comes in length in Section 4. Here it suffices to mention the following points. Firstly, the reduced game may *not* well-defined for *every* S and x. This is not the case for PT reduced games: Any game G has its PT reduced game with respect to any S and x. Their definition differs from mine in how the payoff functions $u_j^{S,x}$ are defined: P&T defines $u_j^{S,x}(y_S) = u_j(y_S, x_{N\setminus S})$ for any $y_S \in X_S$. Secondly, the reduction operation may destroy some properties that the original game possesses (e.g. continuity of the payoff functions). Thus I restrict my attention to families of games within which the reduction operation makes sense.

Definition 2-2. *A family of games $\mathcal{G}$ is said to be* closed *if*

$$\forall G \in \mathcal{G}: \forall S \subset N(G), \quad S \neq \emptyset: \forall x \in X_{N(G)}: G^{S,x} \in \mathcal{G}.$$

This type of closedness appears in P&T together with some other types. Let $\mathcal{G}^*$ denote the family of games $\{G \mid G \text{ is a game, and } \forall j \in N(G)\text{: the payoff}$

[10] I owe this modification of the blocking concept to the associate editor.

function u_j is bounded}. The family $\mathscr{G}^*$ is one example of closed families of games. In Section 3, where I state my theorems, I deal with only all closed subfamilies of $\mathscr{G}^*$ as the domains of the solutions to be axiomatized[11]. I chose these families because in many important economic applications, payoff functions are assumed to be bounded.

Definition 2-3. *Let $\mathscr{G}$ be a family of games. A* solution *on $\mathscr{G}$ is a correspondence $\varphi: \mathscr{G} \rightarrow\rightarrow \bigcup_{G \in \mathscr{G}} X_{N(G)}$ such that $\forall G \in \mathscr{G}: \varphi(G) \subset X_{N(G)}$.*

Definition 2-4. *Let $\mathscr{G}$ be a closed family of games. Let φ be a solution on $\mathscr{G}$. Then if $G \in \mathscr{G}$ has no less than 2 players, i.e., $|N(G)| \geq 2$, then define*

$$\varphi^{\sim}(G) = \{x \in X_{N(G)} \mid \forall S \subset N(G), S \neq \emptyset, N(G): x_S \in \varphi(G^{S,x})\}.$$

Now I am ready to introduce the axioms. Consistency comes first.

Definition 2-5. *Let $\mathscr{G}$ be a closed family of games. A solution φ on $\mathscr{G}$ satisfies* consistency *(CONS) if*

$$\forall G \in \mathscr{G}, \quad |N(G)| \geq 2: \varphi(G) \subset \varphi^{\sim}(G).$$

Definition 2-6. *Let $\mathscr{G}$ be a closed family of games. A solution φ on $\mathscr{G}$ satisfies* converse consistency *(COCONS) if*

$$\forall G \in \mathscr{G}, \quad |N(G)| \geq 2: \varphi(G) \supset \varphi^{\sim}(G).$$

Substituting my reduced games with PT reduced games, this axiom COCONS would be the same as COCONS in P&T, and COCONS_A in Peleg, Potters and Tijs (1996). P&T gives an axiomatization of the Nash equilibrium correspondence with their COCONS axiom.

Definition 2-7. *Let G be a game. Then denote*

$$\text{WPO}(G) = \{x \in X_{N(G)} \mid \neg\exists y \in X_{N(G)}: \forall j \in N(G): u_j(y) > u_j(x)\}.$$

$\text{WPO}(G)$ is simply the set of weakly Pareto optimal strategy profiles of game G.

Definition 2-8. *Let $\mathscr{G}$ be a closed family of games. A solution φ on $\mathscr{G}$ satisfies* converse consistency* *(COCONS*) if*

$$\forall G \in \mathscr{G}, \quad |N(G)| \geq 2: \varphi(G) \supset \varphi^{\sim}(G) \cap \text{WPO}(G).$$

COCONS* is a weaker version of COCONS. If I substitute my reduced games with PT reduced games, this axiom COCONS* would be the same as COCONS^3 in P&T, and COCONS_C in Peleg, Potters and Tijs (1996). A

[11] I am grateful to the associate editor for suggesting these families of games.

similar axiom is used in axiomatizations of the Walras correspondence for generalized economies (Van den Nouweland, Peleg and Tijs, 1996).

Definition 2-9. *Let $\mathcal{G}$ be a family of games. A solution φ on $\mathcal{G}$ satisfies* weak Pareto optimality *(WPO) if*

$$\forall G \in \mathcal{G}\colon \varphi(G) \subset \mathrm{WPO}(G).$$

Definition 2-10. *Let $\mathcal{G}$ be a family of games. A solution φ on $\mathcal{G}$ satisfies* one person rationality *(OPR) if*

$$\forall G \in \mathcal{G}, \quad |N(G)| = 1\colon \varphi(G) = \mathrm{WPO}(G).$$

The axiom OPR simply says that the solution φ solves single person optimization problems.

3. The results

This section states characterization results of the α-core and the α-IR solution. As the domains of these two solutions, I consider all closed subfamilies of $\mathcal{G}^*$, the family of games with bounded payoff functions. Note that $\mathcal{G}^*$ itself is a closed family of games. I start with the following useful lemma.

Lemma 3-1. *Let $\mathcal{G}$ be a non-empty closed family of games. Let μ be a solution on $\mathcal{G}$. Then there exists at most one solution φ on $\mathcal{G}$ which satisfies*

(A) $\forall G \in \mathcal{G}\colon \varphi(G) \subset \mu(G)$,
(B) $\forall G \in \mathcal{G}$, $|N(G)| \geq 2\colon \varphi(G) = \mu(G) \cap \varphi^{\sim}(G)$, and
(C) $\forall G \in \mathcal{G}$, $|N(G)| = 1\colon \mu(G) = \varphi(G)$.

Remark 3-2[12]*:* Lemma 3-1 above shall be applied to prove the 'uniqueness' part of the axiomatic characterizations (Lemmas 3-5 and 3-13). Although $\varphi^{\sim}$ is defined with reference to the reduction operation, Lemma 3-1 holds true regardless of the type of reduced games to adopt. This is also the case for Lemmas 3-5 and 3-13. Thus Lemmas 3-1, 3-5 and 3-13 all hold true even if I alternatively adopt PT reduced games or any other.

Proof of Lemma 3-1: Let μ, ψ_1 and ψ_2 be solutions on $\mathcal{G}$. For μ and each $\varphi \in \{\psi_1, \psi_2\}$, assume that the conditions (A), (B) and (C) in the above are satisfied. Suppose that $\psi_1 \neq \psi_2$. Then there must exist a game $H \in \mathcal{G}$ for which (i) $\psi_1(H) \neq \psi_2(H)$, and (ii) $\forall G \in \mathcal{G}\colon |N(G)| < |N(H)| \Rightarrow \psi_1(G) = \psi_2(G)$. Clearly, by (C), I have $|N(H)| \geq 2$. Without loss of generality, I assume that $x \in \psi_1(H)$ and $x \notin \psi_2(H)$. By (A), $x \in \mu(H)$, Then by (B), I have $x \in \psi_1^{\sim}(H)$ and $x \notin \psi_2^{\sim}(H)$. Thus $\exists S \subset N(H)$, $S \neq \emptyset$, $N(H)$: $x_S \in \psi_1(H^{S,x})$ and $x_S \notin \psi_2(H^{S,x})$. Since $|S| < |N(H)|$, this contradicts (ii). ☺

[12] I owe this remark and the proof of Lemma 3-1 to the associate editor.

3-1. The α-core

My main result is as follows:

Theorem A. *Let $\mathscr{G}$ be a non-empty closed subfamily of $\mathscr{G}^*$. A solution φ on $\mathscr{G}$ satisfies CONS, COCONS*, WPO and OPR if and only if φ is the α-core correspondence.*

To prove this theorem, I use the following two lemmas:

Lemma 3-3. *Let $\mathscr{G}$ be a non-empty closed subfamily of $\mathscr{G}^*$. The α-core correspondence C on $\mathscr{G}$ satisfies CONS, COCONS*, WPO and OPR.*

Proof: Let $G \in \mathscr{G}$. Denote the player set $N(G)$ simply by N.

(i) (OPR): trivial.
(ii) (WPO): Let $x \notin \mathrm{WPO}(G)$. Then N blocks x. Thus $x \notin C(G)$.
(iii) (CONS): Suppose that $x \in C(G)$. Let $S \subset N$ and $S \neq \emptyset$. Then consider the reduced game $G^{S,x}$. Suppose that $x_S \notin C(G^{S,x})$. Then I get $\exists T \subset S \subset N, T \neq \emptyset$: $\exists y_T \in X_T$: $\forall j \in T$: $\inf \mathrm{Im}(u_j^{S,x}|_{yT}) > u_j^{S,x}(x_S)$. That is, $\exists T \subset N$, $T \neq \emptyset$: $\exists y_T \in X_T$: $\forall j \in T$: $\inf_{zS\setminus T \in XS\setminus T} \inf_{zN\setminus S \in XN\setminus S} u_j (y_T, z_{S\setminus T}, z_{N\setminus S}) = \inf \mathrm{Im}(u_j|_{yT}) > u_j(x)$. That is, T blocks x in G. This says $x \notin C(G)$, a contradiction.
(iv) (COCONS*): Let $x \in \mathrm{WPO}(G)$ with $x \notin C(G)$. Thus $\exists S \subset N$, $S \neq \emptyset, N$: $\exists y_S \in X_S$: $\forall j \in S$: $\inf \mathrm{Im}(u_j|_{yS}) > u_j(x)$. That is, $\exists S \subset N$, $S \neq \emptyset, N$: $\exists y_S \in X_S$: $\forall j \in S$: $u_j^{S,x}(y_S) > u_j^{S,x}(x_S)$. Thus I have $x_S \notin C(G^{S,x})$, which implies $x \notin C^{\sim}(G)$. Thus I obtain $(x \in \mathrm{WPO}(G)$ and $x \notin C(G)) \Rightarrow x \notin C^{\sim}(G)$. That is, $x \in \mathrm{WPO}(G) \cap C^{\sim}(G) \Rightarrow x \in C(G)$. ☺

Remark 3-4: If I alternatively adopt the 'standard' blocking concept (see (2) of Section 1) to define the α-core, the axiom COCONS* is not necessarily satisfied. (The other three axioms are all satisfied.) However, the two blocking concepts (the 'standard' one and mine) coincide on some closed subfamilies of $\mathscr{G}^*$. For such families, the 'standard' α-core, which is after all the same as my α-core, is characterized by the present axioms. An example of such families is any closed family $\mathscr{G}$ satisfying the following condition: $\forall G \in \mathscr{G}$: $\forall S \subset N(G)$: $\forall y_S \in X_S$: $\forall j \in S$: $\min \mathrm{Im}(u_j|_{yS})$ exists. For example, the family of all games with finite strategy sets satisfies this condition. In general, if I adopted the 'standard' definition of blocking, then the families of games that my axiomatization covers would be more limited.

Lemma 3-5. *Let $\mathscr{G}$ be a non-empty closed subfamily of $\mathscr{G}^*$. There exists at most one solution φ on $\mathscr{G}$ which satisfies CONS, COCONS*, WPO and OPR.*

Proof: Since CONS, COCONS* and WPO are satisfied, it is immediate that $\forall G \in \mathscr{G}$, $|N(G)| \geq 2$: $\varphi(G) = \mathrm{WPO}(G) \cap \varphi^{\sim}(G)$. Let μ be the solution on $\mathscr{G}$ such that $\forall G \in \mathscr{G}$: $\mu(G) = \mathrm{WPO}(G)$. Then applying Lemma 3-1, I have the desired conclusion. ☺

Now Theorem A follows immediately from Lemmas 3-3 and 3-5.

Remark 3-6: If I replace my reduced games with PT reduced games, then these four axioms characterize the strong Nash equilibrium correspondence (Theorem 3.2 in P&T). In other words, the α-core and strong Nash equilibria are axiomatized with the same combination of axioms but under different types of reduced games.

Further, these four axioms are logically independent. The following examples verify this.

Example 3-7. Let φ be a solution on $\mathscr{G}^*$ such that

$$\forall G \in \mathscr{G}^*: \varphi(G) = \emptyset.$$

Then φ satisfies CONS, COCONS* and WPO, but not OPR.

Example 3-8. Let φ be a solution on $\mathscr{G}^*$ such that

$$\forall G \in \mathscr{G}^*: \varphi(G) = \mathrm{WPO}(G).$$

Then φ satisfies COCONS*, OPR and WPO, but not CONS. The following counterexample shows that CONS is not satisfied:

$$H_1: \begin{array}{ccc} & s_2 & t_2 \\ s_1 & (3,0) & (0,3) \\ t_1 & (1,2) & (2,1) \end{array}$$

Here player 1 chooses row while player 2 column. WPO(H_1) is the whole set of strategy profiles. The reduced game $H_1^{\{1\},(s1,s2)}$ is:

$$\begin{array}{cc} s_1 & 3 \\ t_1 & 1 \end{array}$$

Then WPO$(H_1^{\{1\},(s1,s2)}) = \{s_1\}$. Thus φ violates CONS.

Example 3-9. Let φ be the α-IR solution on $\mathscr{G}^*$.

Then φ satisfies CONS, COCONS* and OPR, but not WPO. This follows from Theorem B in the next subsection. The following counterexample shows that WPO is not satisfied:

$$H_2: \begin{array}{ccc} & s_2 & t_2 \\ s_1 & (5,5) & (0,6) \\ t_1 & (6,0) & (2,2) \end{array}$$

The strategy profile (t_1, t_2) belongs to the α-IR solution of H_2 but is Pareto-dominated by (s_1, s_2). Thus φ violates WPO.

Example 3-10. Let φ be a solution on $\mathscr{G}^*$ such that

$$\forall G \in \mathscr{G}^*\colon \varphi(G) = \text{WPO}(G) \quad \text{if } |N(G)| = 1,$$

$$= \varnothing \quad \text{otherwise.}$$

Then φ satisfies CONS, OPR and WPO, but not COCONS*. The following counterexample checks this:

$$F\colon \begin{array}{ccc} & s_2 & t_2 \\ s_1 & (1,4) & (4,2) \\ t_1 & (2,3) & (3,1) \end{array}$$

$\text{WPO}(F) = \{(s_1, s_2), (s_1, t_2), (t_1, s_2)\}$. Consider the reduced games $F^{\{1\},(t1,s2)}$ and $F^{\{2\},(t1,s2)}$.

$$F^{\{1\},(t1,s2)}\colon \begin{array}{cc} s_1 & 1 \\ t_1 & 2 \end{array}$$

$$F^{\{2\},(t1,s2)}\colon \begin{array}{cc} s_2 & t_2 \\ 3 & 1 \end{array}$$

Then $t_1 \in \text{WPO}(F^{\{1\},(t1,s2)})$ and $s_2 \in \text{WPO}(F^{\{2\},(t1,s2)})$. Thus $(t_1, s_2) \in \varphi^{\sim}(F)$. Then I get $\varphi^{\sim}(F) \cap \text{WPO}(F) \neq \varnothing$. Thus φ violates COCONS*.

3-2. The α-individually rational solution

In Example 3-9, I indicated that the α-IR solution satisfies CONS, COCONS* and OPR. To verify this, I axiomatize the α-IR solution.

Theorem B. *Let $\mathscr{G}$ be a non-empty closed subfamily of $\mathscr{G}^*$. A solution φ on $\mathscr{G}$ satisfies CONS, COCONS and OPR if and only if φ is the α-IR solution correspondence.*

Remark 3-11: Comparing with the foregoing axiomatization of the α-core, this system of axioms not only drops WPO but adopts a stronger version of converse consistency (COCONS implies COCONS*).

The proof of Theorem B is similar to the proof of Theorem A.

Lemma 3-12. *Let $\mathscr{G}$ be a non-empty closed subfamily of $\mathscr{G}^*$. The α-IR solution IR on $\mathscr{G}$ satisfies CONS, COCONS and OPR.*

Proof: Let $G \in \mathscr{G}$. Denote the player set $N(G)$ simply by N.

(i) (OPR): trivial.
(ii) (CONS): Let $x \in \mathrm{IR}(G)$. Suppose that $\exists S \subset N,\ S \neq \emptyset,\ N$: $x_S \notin \mathrm{IR}(G^{S,x})$. That is, $\exists j \in S$: $\{j\}$ blocks x_S in $G^{S,x}$. Then setting $T = \{j\}$, the same argument as in (iii) of the proof of Lemma 3-3 applies. Then I have that $\{j\}$ blocks x in G. That is, $x \notin \mathrm{IR}(G)$, a contradiction.
(iii) (COCONS): Suppose that $x \in X_N$ and $x \notin \mathrm{IR}(G)$. That is, $\exists j \in N$: $\{j\}$ blocks x in G. Then, setting $S = \{j\}$, applying the same argument as in (iv) of the proof of Lemma 3-3, I have that $\{j\}$ blocks x_j in $G^{\{j\},x}$. That is, $x_j \notin \mathrm{IR}(G^{\{j\},x})$. The desired conclusion follows. ☺

Lemma 3-13. *Let $\mathcal{G}$ be a non-empty closed subfamily of $\mathcal{G}^*$. There exists at most one solution φ on $\mathcal{G}$ which satisfies CONS, COCONS and OPR.*

Proof: It is immediate from CONS and COCONS that $\forall G \in \mathcal{G}$, $|N(G)| \geq 2$: $\varphi^{\sim}(G) = \varphi(G)$. Let μ be the solution on $\mathcal{G}$ such that $\forall G \in \mathcal{G}$: $|N(G)| \geq 2 \Rightarrow \mu(G) = X_{N(G)}$, and $|N(G)| = 1 \Rightarrow \mu(G) = \mathrm{WPO}(G)$. Then applying Lemma 3-1, I obtain the desired conclusion. ☺

Now Theorem B follows directly from Lemmas 3-12 and 3-13.

Remark 3-14: Substituting my reduced games with PT reduced games, these three axioms give an axiomatization of the Nash equilibrium correspondence (Theorem 2.12 in P&T). This is analogous to the fact mentioned in Remark 3-6. Readers can find some discussion on this point in Section 4.

One can check the logical independence of these three axioms by the following examples:

Example 3-15. Let φ be a solution on $\mathcal{G}^*$ such that

$$\forall G \in \mathcal{G}^*: \varphi(G) = \emptyset.$$

Then φ satisfies CONS and COCONS, but not OPR.

Example 3-16. Let φ be a solution on $\mathcal{G}^*$ such that

$$\forall G \in \mathcal{G}^*: \varphi(G) = \mathrm{WPO}(G) \quad \text{if } |N(G)| = 1,$$

$$= \emptyset \quad \text{otherwise.}$$

Then φ satisfies CONS and OPR, but not COCONS.

Example 3-17. Let φ be a solution on $\mathcal{G}^*$ such that

$$\forall G \in \mathcal{G}^*: \varphi(G) = \mathrm{WPO}(G) \quad \text{if } |N(G)| = 1,$$

$$= X_{N(G)} \quad \text{otherwise.}$$

Then φ satisfies COCONS and OPR, but not CONS.

4. Discussion

This section discusses the following two points: (I) On the interpretations of reduced games. (II) An implication of the facts indicated in Remarks 3-6 and 3-14.

(I)[13] Although formally defined, I have not yet given an intuitive interpretation of my reduced games. On the other hand, PT reduced games have a straightforward interpretation. They remarked: "(Let a game G be given and $\emptyset \neq S \subset N(G)$ and $x \in X_{N(G)}$.) If it is common knowledge among the members of S that the members of $N(G) \backslash S$ have chosen the strategies x_i, $i \in N(G) \backslash S$, then the members of S are faced with the game $G^{S,x}$." Although not so simple as this one, the following interpretation of my reduced games seems to illustrate the point. Let a game G be given, and let $\emptyset \neq S \subset N(G)$ and $x \in X_{N(G)}$. Now split the game G into two stages. In the first stage, the members of S choose their strategies, and then, in the second stage, the members of $N(G) \backslash S$ move in response knowing what S has chosen. Each player belonging to S *believes* that (i) $N(G) \backslash S$ will play $x_{N(G) \backslash S}$ if S plays x_S, and that (ii) if S choose $y_S \neq x_S$, then $N(G) \backslash S$ will jointly punish *him* by practising the strategies that minimizes *his* payoff (given y_S has been chosen). Then, in effect, the first stage of the game is equivalent to the reduced game $G^{S,x}$.

To be fair, however, I have to point out the following difficulties about this interpretation.

(a) The members of S do *not* expect that, in the second stage, the players in $N(G) \backslash S$ behave as payoff maximizers when S deviated from the strategy profile x_S.
(b) It is not always possible for $N(G) \backslash S$ to jointly choose strategies to simultaneously minimize the payoff of each member of S.
(c) Each player j in S believes that, when S *collectively* deviated from x_S, *he* is to be punished regardless of whether or not *he* deviated from *his own* strategy x_j.

Despite all these facts, I am assuming that the members of S believe (ii) in my interpretation. I consider this to be an expression of the 'pessimism[14]' found in the 'maximin' thought, which underlies both the α-IR solution and the α-core. My scenario reflects the nature of the solutions to be axiomatized. This makes a clear contrast to the idea behind PT reduced games. In the PT reduced game $G^{S,x}$, a player in S is 'passive', rather than 'pessimistic', in a sense, to the choices of strategies by the members of $N(G) \backslash S$. Or one might see that in the

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[13] I thank the associate editor for various comments which were useful in improving this part of the paper.

[14] This word expresses the point that both in the α-core and the α-IR solution, a coalition (a player) is concerned with every conceivable action open to the complementary coalition regardless of its likelihood. Scarf (1971) pointed out this as a drawback of the α-core: "A coalition S, in attempting to obtain an improved position for all of its members must confront the entire range of strategic possibilities open to the players not in S, including those which lead to disastrous consequence for the complementary coalition and would in all probability not be undertaken. This inability to discriminate among counterresponses … results in the inclusion of more outcomes in the solution than might seem reasonable."

case of the PT reduced game $G^{S,x}$, a player in S has 'more information' about the choices to be made by the players in $N(G)\backslash S$ than in the case of my reduced game.

However, there may be an alternative path to avoid the difficulty (c). One may define reduced games as follows:

$$\text{(ARG)} \quad u_j^{S,x}(y_S) = u_j(x) \quad \text{if } y_j = x_j,$$

$$= \inf \operatorname{Im}(u_j|_{yS}) \quad \text{otherwise.}$$

According to the definition (ARG) in the above, *only* player(s) j who deviated from the strategy x_j is to be punished in the corresponding scenario. Interestingly, it is not difficult to prove that both the α-core and the α-IR solution satisfy consistency under the definition (ARG) (proof in Appendix). Further, as far as the reduced games with respect to any *singleton coalitions* (i.e., S is a singleton in the above) are considered, the definition (ARG) is the same as Definition 2-1. In the proof that the α-IR solution satisfies COCONS (Lemma 3-12, (iii)), I considered only one-person reduced games. Then it is immediate that the α-IR solution satisfies COCONS also under the definition (ARG). Lemma 3-13 also applies without any change (see Remark 3-2). Thus I conclude that the characterization of the α-IR solution (Theorem B) holds true even *under the definition of reduced games (ARG)*. However, at the present time, I am not able to axiomatize the α-core using the definition (ARG). It is a subject of future research to see whether this is possible or not.

(II) As pointed out in Remarks 3-6 and 3-14, the two pairs of solutions, {the α-core, strong Nash equilibria} and {the α-IR solution, Nash equilibria}, are respectively axiomatized by the same system of axioms under different definitions of reduced games[15]. The α-core of a game is the set of strategy profiles that *no coalition* blocks. And the α-IR solution is the set of strategy profiles that *no single player* blocks. Similarly, a strong Nash equilibrium is a strategy profile from which *no coalition* can deviate with each member of the coalition improving his payoff. And a Nash equilibrium is a strategy profile from which *no single player* can profitably deviate. One may see a similarity between the two pairs of solutions,{the α-core, the α-IR solution} and {strong Nash equilibria, Nash equilibria}, in the way one solution relates to the other. The points made in Remarks 3-6 and 3-14 give a formal content to this intuitive similarity. I find this rather important from the viewpoint that one of the main purposes of axiomatizations is to *compare solutions*.

Appendix

The following claim is established *under the definition of reduced games (ARG)* in Section 4(I) in the place of Definition 2-1.

[15] Interestingly, a similar thing happens in coalitional form games: The Shapley value and the prenucleolus are axiomatized by the same axioms under different types of reduced games (Hart and Mas-Colell, 1989 and Sobolev, 1975). Maschler (1990) stated: "I find this a fascinating result: It shows that the intrinsic difference between the Shapley value and the prenucleolus lies in the way the subsets S of N interpret 'their own game'!"

Claim: Let $\mathscr{G}$ be a non-empty closed subfamily of $\mathscr{G}^*$. Then both the α-core correspondence C and the α-IR solution correspondence IR on $\mathscr{G}$ satisfy CONS.

Proof: (α-core): Let $G = (N, \{X_j, u_j\}_{j \in N}) \in \mathscr{G}$. Assume that $x \in C(G)$. Let $S \subset N$ with $S \neq \emptyset, N$. Suppose that $x_S \notin C(G^{S,x})$. Then $\exists T \subset S \subset N$, $T \neq \emptyset$: $\exists y_T \in X_T$: $\forall j \in T$: $\inf \mathrm{Im}(u_j^{S,x}|_{y_T}) > u_j^{S,x}(x_S)$. Denote by U the set $\{j \in T \mid y_j \neq x_j\}$. Clearly, $U \neq \emptyset$. Then by the definition (ARG), $\forall j \in U$: $\inf_{z_{S \setminus T} \in X_{S \setminus T}} \inf_{z_{N \setminus S} \in X_{N \setminus S}} u_j(y_T, z_{S \setminus T}, z_{N \setminus S}) > u_j(x)$ and $\forall j \in T \setminus U$: $u_j(x) > u_j(x)$. This implies $U = T$. Thus I have $\forall j \in T$: $\inf \mathrm{Im}(u_j|_{y_T}) > u_j(x)$. That is, T blocks x in G. This says $x \notin C(G)$, a contradiction.

(α-IR solution): Similar (Set $T = \{j\}$ in the above proof). ☺

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